The method of brackets. Part 2: examples and applications

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ABSTRACT. A new heuristic method for the evaluation of definite integrals is presented. This *method of brackets* has its origin in methods developed for the evaluation of Feynman diagrams. The operational rules are described and the method is illustrated with several examples. The method of brackets reduces the evaluation of a large class of definite integrals to the solution of a linear system of equations.

1. Introduction

The *method of brackets* presented here provides a method for the evaluation of a large class of definite integrals. The ideas were originally presented in [**6**] in the context of integrals arising from Feynman diagrams. A complete description of the operational rules of the method together with a variety of examples was first discussed in [**5**].

The method is quite simple to work with and many of the entries from the classical table of integrals [**7**] can be derived using this method. The basic idea is to introduce the formal symbol $\langle a \rangle$, called a *bracket*, which represents the divergent integral

$$(1.1) \qquad \int_0^\infty x^{a-1} \, dx.$$

The formal rules for operating with these brackets are described in Section 2 and their justification (especially of the heuristic Rule 2.3) is work-in-progress. In particular, convergence issues are ignored at the moment. Roughly, each integral generates a linear system of equations and for each choice of free variables the method yields a series with the free variables as summation indices. A heuristic rule states that those converging in a common region give the desired evaluation.

Section 3 illustrates the method by evaluating the Laplace transform of the Bessel function $J_\nu(x)$. In this example, the two resulting series converge in different regions and are analytic continuations of each other. This is a general phenomenon which is used in Section 5 to produce an explicit analytic continuation of the hypergeometric function $_{q+1}F_q(x)$. Section 4 presents the evaluation of a family of integrals C_n appearing in Statistical Mechanics. These were introduced in [**4**] as a

toy model and their physical interpretation was discovered later. The method of brackets is employed here to evaluate the first four values, the only known cases (an expression for the next value C_5 in terms of a double hypergeometric series is possible but is not given here). The last section employs the method of brackets to resolve a Feynman diagram.

2. The method of brackets

The method of brackets discussed in this paper is based on the assignment of the formal symbol $\langle a \rangle$ to the divergent integral (1.1).

EXAMPLE 2.1. If f is given by the formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},$$

then the improper integral of f over the positive real axis is formally written as the *bracket series*

$$(2.1) \qquad \int_0^{\infty} f(x)\, dx = \sum_n a_n \langle \alpha n + \beta \rangle.$$

Here, and in the sequel, $\sum_n$ is used as a shorthand for $\sum_{n=0}^{\infty}$.

Formal rules for operating with brackets are described next. In particular, Rule 2.2 describes how to evaluate a bracket series such as the one appearing in (2.1). To this end, it is useful to introduce the symbol

$$(2.2) \qquad \phi_n = \frac{(-1)^n}{\Gamma(n+1)},$$

which is called the *indicator of n*.

EXAMPLE 2.2. The gamma function has the bracket expansion

$$(2.3) \qquad \Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x}\, dx = \sum_n \phi_n \langle n + a \rangle.$$

RULE 2.1. The bracket expansion

$$(2.4) \quad \frac{1}{(a_1 + a_2 + \cdots + a_r)^{\alpha}} = \sum_{m_1, \ldots, m_r} \phi_{m_1, \ldots, m_r} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle \alpha + m_1 + \cdots + m_r \rangle}{\Gamma(\alpha)}$$

holds. Here $\phi_{m_1, \ldots, m_r}$ is a shorthand notation for the product $\phi_{m_1} \cdots \phi_{m_r}$. If there is no possibility of confusion this will be further abridged as $\phi_{\{m\}}$. The notation $\sum_{\{m\}}$ is to be understood likewise.

RULE 2.2. A series of brackets is assigned a value according to

$$(2.5) \qquad \sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*),$$

where n^* is the solution of the equation $an + b = 0$. Observe that this might result in the replacing of the index n, initially a nonnegative integer, by a complex number n^*.

Similarly, a higher dimensional bracket series, that is,

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1, \ldots, n_r) \langle a_{11} n_1 + \cdots a_{1r} n_r + c_1 \rangle \cdots \langle a_{r1} n_1 + \cdots a_{rr} n_r + c_r \rangle$$

is assigned the value

$$(2.6) \qquad \frac{1}{|\det(A)|} f(n_1^*, \cdots, n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*),$$

where A is the matrix of coefficients (a_{ij}) and (n_i^*) is the solution of the linear system obtained by the vanishing of the brackets. The value is not defined if the matrix A is not invertible.

RULE 2.3. In the case where a higher dimensional series has more summation indices than brackets, the appropriate number of free variables is chosen among the indices. For each such choice, Rule 2.2 yields a series. Those converging in a common region are added to evaluate the desired integral.

3. An example from Gradshteyn and Ryzhik

The second author is involved in a long term project of providing proofs of all the entries from the classical table of integrals by Gradshteyn and Ryzhik [7]. The proofs can be found at:

http://www.math.tulane.edu/~vhm/Table.html

In this section the method of brackets is illustrated to find

$$(3.1) \qquad \int_0^\infty x^\nu e^{-\alpha x} J_\nu(\beta x)\, dx = \frac{(2\beta)^\nu \Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}(\alpha^2 + \beta^2)^{\nu+1/2}}$$

which is entry 6.623.1 of [7]. Here

$$(3.2) \qquad J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{2k+\nu}}{k!\, \Gamma(k+\nu+1)}$$

is the Bessel function of order ν. To this end, the integrand is expanded as

$$(3.3) \qquad e^{-\alpha x} J_\nu(\beta x) = \left(\sum_n \phi_n (\alpha x)^n \right) \left(\sum_k \phi_k \frac{(\frac{\beta x}{2})^{2k+\nu}}{\Gamma(k+\nu+1)} \right)$$

$$= \sum_{k,n} \phi_{k,n} \frac{\alpha^n (\frac{\beta}{2})^{2k+\nu}}{\Gamma(k+\nu+1)} x^{n+2k+2\nu},$$

so as to obtain the bracket series

$$(3.4) \qquad \int_0^\infty e^{-\alpha x} J_\nu(\beta x)\, dx = \sum_{k,n} \phi_{k,n} \frac{\alpha^n (\frac{\beta}{2})^{2k+\nu}}{\Gamma(k+\nu+1)} \langle n + 2k + 2\nu + 1 \rangle.$$

The evaluation of this double sum by the method of brackets produces two series corresponding to using either k or n as the free variable when applying Rule 2.2.

The index k is free. Choosing k as the free variable when applying Rule 2.2 to (3.4), yields $n^* = -2k - 2\nu - 1$ and thus the resulting series

$$(3.5) \qquad \sum_k \phi_k \frac{\alpha^{-2k-2\nu-1}(\frac{\beta}{2})^{2k+\nu}}{\Gamma(k+\nu+1)} \Gamma(2k+2\nu+1)$$

$$= \alpha^{-2\nu-1}(\tfrac{\beta}{2})^\nu \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}\, {}_1F_0 \left(\begin{matrix} \nu + \frac{1}{2} \\ - \end{matrix} \middle| -\frac{\beta^2}{\alpha^2} \right).$$

The right-hand side employs the usual notation for the hypergeometric function

$$(3.6) \qquad {}_pF_q \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is the Pochhammer symbol. Note that the ${}_1F_0$ in (3.5) converges provided $|\beta| < |\alpha|$. In this case, the standard identity ${}_1F_0(a|x) = (1 - x)^{-a}$ together with the duplication formula for the Γ function shows that the series in (3.5) is indeed equal to the right-hand side of (3.1).

The index n is free. In this second case, the linear system in Rule 2.2 has determinant 2 and yields $k^* = -n/2 - \nu - 1/2$. This gives

$$(3.7) \qquad \frac{1}{2} \sum_n \phi_n \frac{\alpha^n (\frac{\beta}{2})^{-n-\nu-1}}{\Gamma(-n/2 + 1/2)} \Gamma(n/2 + \nu + 1/2).$$

This series now converges provided that $|\beta| > |\alpha|$ in which case it again sums to the right-hand side of (3.1).

NOTE. *This is the typical behavior of the method of brackets. The different choices of indices as free variables give representations of the solution valid in different regions. Each of these is an analytic continuation of the other ones.*

4. Integrals of the Ising class

In this section the method of brackets is used to discuss the integral

$$(4.1) \qquad C_n = \frac{4}{n!} \int_0^{\infty} \cdots \int_0^{\infty} \frac{1}{\left(\sum_{j=1}^{n} (u_j + 1/u_j) \right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}.$$

This family was introduced in [4] as a caricature of the *Ising susceptibility integrals*

$$(4.2) \qquad D_n = \frac{4}{n!} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i<j} \left(\frac{u_i - u_j}{u_i + u_j} \right)^2 \frac{1}{\left(\sum_{j=1}^{n} (u_j + 1/u_j) \right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}.$$

Actually, the integrals C_n appear naturally in the analysis of certain amplitude transforms [10]. The first few values are given by

$$(4.3) \qquad C_1 = 2, \ C_2 = 1, \ C_3 = L_{-3}(2), \ C_4 = \frac{7}{12} \zeta(3).$$

Here, L_D is the Dirichlet L-function. In this case,

$$(4.4) \qquad L_{-3}(2) = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right).$$

No analytic expression for C_n is known for $n \geq 5$. Similarly,

$$(4.5) \qquad D_1 = 2, \ D_2 = \frac{1}{3}, \ D_3 = 8 + \frac{4\pi^2}{3} - 27 L_{-3}(2), \ D_4 = \frac{4\pi^2}{9} - \frac{1}{6} - \frac{7}{12} \zeta(3)$$

are given in [4]. High precision numerical evaluation and PSLQ experiments have further produced the conjecture

$$(4.6) \qquad D_5 = 42 - 1984 \text{Li}_4(\tfrac{1}{2}) + \frac{189}{10} \pi^4 - 74\zeta(3) - 1272\zeta(3) \ln 2 + 40\pi^2 \ln^2 2$$

$$- \frac{62}{3} \pi^3 + \frac{40}{3} \pi^2 \ln 2 + 88 \ln^4 2 + 464 \ln^2 2 - 40 \ln 2.$$

The integral C_n is the special case $k = 1$ of the family

$$(4.7) \qquad C_{n,k} = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^{k+1}} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

that also gives the moments of powers of the Bessel function K_0 via

$$(4.8) \qquad C_{n,k} = \frac{2^{n-k+1}}{n!\,k!} c_{n,k} := \frac{2^{n-k+1}}{n!\,k!} \int_0^\infty t^k K_0^n(t)\, dt.$$

The values

$$(4.9) \qquad c_{1,k} = 2^{k-1}\Gamma^2\left(\frac{k+1}{2}\right), \quad c_{2,k} = \frac{\sqrt{\pi}}{4} \frac{\Gamma^3\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)},$$

as well as the recursion

$$(4.10) \qquad (k+1)^4 c_{3,k} - 2(5k^2 + 20k + 21)c_{3,k+2} + 9c_{3,k+4} = 0$$

with initial data

$$(4.11) \qquad c_{3,0} = \frac{3\alpha}{32\pi},\ c_{3,1} = \frac{3}{4}L_{-3}(2),\ c_{3,2} = \frac{\alpha}{96\pi} - \frac{4\pi^5}{9\alpha},\ c_{3,3} = L_{-3}(2) - \frac{2}{3},$$

where $\alpha = 2^{-2/3}\Gamma^6(\frac{1}{3})$ are given in [**1**] and [**3**].

The evaluation of these integrals presented in the literature usually begins with the introduction of spherical coordinates. This reduces the dimension of C_n by two and immediately gives the values of C_1 and C_2. The evaluation of C_3 is reduced to the logarithmic integral

$$(4.12) \qquad C_3 = \frac{2}{3} \int_0^\infty \frac{\ln(1+x)\, dx}{x^2 + x + 1}.$$

Its value is obtained by the change of variables $x \to \frac{1}{t} - 1$ followed by an expansion of the integrand. A systematic discussion of these type of logarithmic integrals is provided in [**9**]. The value of C_4 is obtained via the double integral representation

$$(4.13) \qquad C_4 = \frac{1}{6} \int_0^\infty \int_0^\infty \frac{\ln(1+x+y)}{(1+x+y)(1+1/x+1/y) - 1} \frac{dx}{x} \frac{dy}{y}.$$

Moreover, the limiting behavior

$$(4.14) \qquad \lim_{n\to\infty} C_n = 2e^{-2\gamma}$$

was established in [**4**].

In this section the method of brackets is used to obtain the expressions for C_2, C_3, and C_4 described above. An advantage of this method is that it systematically gives an analytic expression for these integrals. When applied to C_5, the method produces a double series representation which is not discussed here.

4.1. Evaluation of $C_{2,k}$. The numbers $C_{2,k}$ are given by

$$(4.15) \qquad C_{2,k} = 2 \int_0^\infty \int_0^\infty \frac{dx\, dy}{xy\,(x+1/x+y+1/y)^{k+1}}.$$

A direct application of the method of brackets, by applying Rule 2.1 to the integrand as in (4.15), results in a bracket expansion involving a 4-fold sum and 3 brackets. Rules 2.2 and 2.3 translates this into a collection of series with $4 - 3 = 1$

summation indices. However, it is generally desirable to minimize the final number of summations by reducing the number of sums and increasing the number of brackets. In this example this is achieved by writing

$$C_{2,k} = 2 \int_0^\infty \int_0^\infty \frac{(xy)^k \, dx \, dy}{(x^2 y + y + xy^2 + x)^{k+1}}$$

$$= 2 \int_0^\infty \int_0^\infty \frac{(xy)^k \, dx \, dy}{(xy \, [x+y] + [x+y])^{k+1}}.$$

In the evaluation of these expressions, the term $(x+y)$ must be expanded at the last step. The method of brackets now yields

$$\frac{1}{(xy \, [x+y] + [x+y])^{k+1}} = \sum_{n_1, n_2} \phi_{n_1, n_2} \, x^{n_1} y^{n_1} \, (x+y)^{n_1 + n_2} \frac{\langle k+1+n_1+n_2 \rangle}{\Gamma(k+1)},$$

and the expansion of the term $(x+y)$ gives

$$\frac{1}{(x+y)^{-n_1 - n_2}} = \sum_{n_3, n_4} \phi_{n_3, n_4} \, x^{n_3} y^{n_4} \frac{\langle -n_1 - n_2 + n_3 + n_4 \rangle}{\Gamma(-n_1 - n_2)}.$$

Replacing in the integral produces the bracket expansion

$$C_{2,k} = 2 \sum_{\{n\}} \phi_{\{n\}} \frac{\langle k+1+n_1+n_2 \rangle}{\Gamma(k+1)} \frac{\langle -n_1 - n_2 + n_3 + n_4 \rangle}{\Gamma(-n_1 - n_2)}$$

$$\times \langle k+1+n_1+n_3 \rangle \langle k+1+n_1+n_4 \rangle.$$

The value of this formal sum is now obtained by solving the linear system $k+1+n_1+n_2 = 0$, $-n_1 - n_2 + n_3 + n_4 = 0$, $k+1+n_1+n_3 = 0$, and $k+1+n_1+n_4 = 0$ coming from the vanishing of brackets. This system has determinant 2 and its unique solution is $n_1^* = n_2^* = n_3^* = n_4^* = -\frac{k+1}{2}$. It follows that

$$(4.16) \qquad C_{2,k} = \frac{\Gamma(-n_1^*) \Gamma(-n_2^*) \Gamma(-n_3^*) \Gamma(-n_4^*)}{\Gamma(k+1) \Gamma(-n_1^* - n_2^*)} = \frac{\Gamma\left(\frac{k+1}{2}\right)^4}{\Gamma(k+1)^2}.$$

Note that, upon employing Legendre's duplication formula for the Γ function, this evaluation is equivalent to (4.9). In particular, this confirms the value $C_2 = C_{2,1} = 1$ in (4.3).

REMARK 4.1. The evaluation

$$(4.17) \qquad C_{2,k}(\alpha, \beta) = 2 \int_0^\infty \int_0^\infty \frac{x^{\alpha-1} y^{\beta-1} \, dx \, dy}{(x + 1/x + y + 1/y)^{k+1}}$$

$$= \frac{\Gamma\left(\frac{k+1+\alpha+\beta}{2}\right) \Gamma\left(\frac{k+1-\alpha-\beta}{2}\right) \Gamma\left(\frac{k+1+\alpha-\beta}{2}\right) \Gamma\left(\frac{k+1-\alpha+\beta}{2}\right)}{\Gamma(k+1)^2}$$

that generalizes $C_{2,k}$ is obtained as a bonus. Similarly,

$$(4.18) \qquad J_{r,s}(\alpha, \beta) = 2 \int_0^\infty \int_0^\infty \frac{x^{\alpha-1} y^{\beta-1} \, dx \, dy}{(x+y)^r (xy+1)^s}$$

$$= \frac{\Gamma\left(\frac{-r+\alpha+\beta}{2}\right) \Gamma\left(\frac{2s+r-\alpha-\beta}{2}\right) \Gamma\left(\frac{r+\alpha-\beta}{2}\right) \Gamma\left(\frac{r-\alpha+\beta}{2}\right)}{\Gamma(r) \Gamma(s)}.$$

Note that $C_{2,k}(\alpha, \beta) = J_{k+1,k+1}(\alpha+k+1, \beta+k+1)$.

REMARK 4.2. The Ising susceptibility integral D_2, see (4.2), is obtained directly from the expression for $J_{r,s}$ given above. Indeed,

$$(4.19) \qquad D_2 = 2 \int_0^\infty \int_0^\infty (x^2 - 2xy + y^2) \frac{xy \, dx \, dy}{(x+y)^4 (xy+1)^2}$$
$$= 2 \left(J_{4,2}(4,2) - 2 J_{4,2}(3,3) + J_{4,2}(2,4) \right)$$
$$= \frac{1}{3}.$$

This agrees with (4.5). This technique also yields the generalization

$$(4.20) \qquad D_2(\alpha, \beta) = 2 \int_0^\infty \int_0^\infty \left(\frac{x-y}{x+y} \right)^2 \frac{x^{\alpha-1} y^{\beta-1} \, dx \, dy}{(x + 1/x + y + 1/y)^2}$$
$$= \frac{(b-a)(b+a)(2 + (b-a)^2)\pi^2}{12(\cos(\alpha\pi) - \cos(\beta\pi))}$$

with limiting case $D_2(\alpha, \alpha) = \frac{1}{3} \frac{\alpha\pi}{\sin(\alpha\pi)}$.

4.2. Evaluation of $C_{3,k}$. Next, consider the integral

$$(4.21) \qquad C_{3,k} = \frac{2}{3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{dx \, dy \, dz}{xyz \, (x + 1/x + y + 1/y + z + 1/z)^{k+1}}$$
$$= \frac{2}{3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{(xyz)^k \, dx \, dy \, dz}{(xyz \, (x+y) + z \, (x+y) + xyz^2 + xy)^{k+1}}.$$

The second form of the integrand is motivated by the desire to to minimize the number of sums and to maximize the number of brackets in the expansion. The denominator is now expanded as

$$\sum_{\{n\}} \phi_{\{n\}} (xy)^{n_1 + n_3 + n_4} z^{n_1 + n_2 + 2n_3} (x+y)^{n_1+n_2} \frac{\langle k + 1 + n_1 + n_2 + n_3 + n_4 \rangle}{\Gamma(k+1)},$$

and further expanding $(x+y)^{n_1+n_2}$ as

$$(x+y)^{n_1+n_2} = \sum_{n_5, n_6} \phi_{n_5, n_6} \, x^{n_5} y^{n_6} \frac{\langle -n_1 - n_2 + n_5 + n_6 \rangle}{\Gamma(-n_1 - n_2)}$$

produces a complete bracket expansion of the integrand of $C_{3,k}$. Integration then yields

$$(4.22) \qquad C_{3,k} = \frac{2}{3} \frac{1}{k!} \sum_{\{n\}} \phi_{\{n\}} \frac{\langle -n_1 - n_2 + n_5 + n_6 \rangle}{\Gamma(-n_1 - n_2)}$$
$$\times \langle k + 1 + n_1 + n_2 + n_3 + n_4 \rangle \langle k + 1 + n_1 + n_3 + n_4 + n_5 \rangle$$
$$\times \langle k + 1 + n_1 + n_3 + n_4 + n_6 \rangle \langle k + 1 + n_1 + n_2 + 2n_3 \rangle.$$

This expression is regularized by replacing the bracket $\langle k + 1 + n_1 + n_2 + 2n_3 \rangle$ with $\langle k + 1 + n_1 + n_2 + 2n_3 + \epsilon \rangle$ with the intent of letting $\epsilon \to 0$. (This corresponds to multiplying the initial integrand with z^ϵ; however, note that many other regularizations are possible and eventually lead to Theorem 4.3. It will become clear shortly, see (4.24), why regularizing is necessary.) The method of brackets now gives a set of series expansions obtained by the vanishing of the five brackets in (4.22). The solution of the corresponding linear system (which has determinant 2) leaves one free index and produces the integral as a series in this variable. Of the six possible

free indices, only n_3 and n_4 produce convergent series (more specifically, for each free index one obtains a hypergeometric series ${}_3F_2$ times an expression free of the index; for the indices n_3, n_4 the argument of this ${}_3F_2$ is $\frac{1}{4}$ while otherwise it is 4.) The heuristic Rule 2.3 states that their sum yields the value of the integral:

$$(4.23) \qquad C_{3,k} = \frac{1}{3} \lim_{\epsilon \to 0} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (f_{k,n}(\epsilon) + f_{k,n}(-\epsilon))$$

where

$$(4.24) \qquad f_{k,n}(\epsilon) = \frac{\Gamma\left(n + \frac{k+1+\epsilon}{2}\right)^4 \Gamma(-n-\epsilon)}{\Gamma(2n + k + 1 + \epsilon)}.$$

Observe that the terms $f_{k,n}(\epsilon)$ are contributed by the index n_3 while the terms $f_{k,n}(-\epsilon)$ come from the index n_4. At $\epsilon = 0$, each of them has a simple pole. Consequently, the even combination $f_{k,n}(\epsilon) + f_{k,n}(-\epsilon)$ has no pole at $\epsilon = 0$. Using the expansions

$$(4.25) \qquad \Gamma(x + \epsilon) = \Gamma(x)(1 + \psi(x)\epsilon) + O(\epsilon^2),$$

for $x \neq 0, -1, -2, \ldots$, as well as

$$(4.26) \qquad \Gamma(-n + \epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} + \psi(n+1)\right) + O(\epsilon),$$

for $n = 0, 1, 2, \ldots$, provides the next result.

THEOREM 4.3. *The integrals $C_{3,k}$ are given by*

$$C_{3,k} = \frac{2}{3} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \frac{\Gamma\left(n + \frac{k+1}{2}\right)^4}{\Gamma(2n+k+1)} \left(\psi(n+1) - 2\psi\left(n + \tfrac{k+1}{2}\right) + \psi(2n + k + 1)\right).$$

In particular, for $k = 1$

$$(4.27) \qquad C_3 = \frac{2}{3} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)!} \left(\psi(2n+2) - \psi(n+1)\right).$$

The evaluation of this sum using `Mathematica 7` yields a large collection of special values of (poly-)logarithms. After simplifications, it yields $C_3 = L_{-3}(2)$ as in (4.3).

REMARK 4.4. An extension of Theorem 4.3 is presented next:

$$(4.28) \qquad C_{3,k}(\alpha, \beta, \gamma) = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{x^{\alpha-1} y^{\beta-1} z^{\gamma-1} \, dx \, dy \, dz}{(x + 1/x + y + 1/y + z + 1/z)^{k+1}},$$

for $\gamma = 0$, is given by

$$\frac{1}{k!} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \frac{\Gamma\left(n + \frac{k+1\pm\alpha\pm\beta}{2}\right)}{\Gamma(2n+k+1)} \left(\psi(n+1) - \frac{1}{2}\psi\left(n + \frac{k+1\pm\alpha\pm\beta}{2}\right) + \psi(2n+k+1)\right)$$

where the notation $\Gamma(n + \frac{k+1\pm\alpha\pm\beta}{2}) = \Gamma(n + \frac{k+1+\alpha+\beta}{2})\Gamma(n + \frac{k+1+\alpha-\beta}{2})\cdots$ as well as $\psi(n + \frac{k+1\pm\alpha\pm\beta}{2}) = \psi(n + \frac{k+1+\alpha+\beta}{2}) + \psi(n + \frac{k+1+\alpha-\beta}{2}) + \cdots$ is employed. Similar expressions can be given for other integral values of γ. In the case where γ is not integral, $C_{3,k}(\alpha, \beta, \gamma)$ can be written as a sum of two ${}_3F_2$'s with Γ factors. The symmetry of $C_{3,k}(\alpha, \beta, \gamma)$ in α, β, γ, shows that this can be done if at least one of these arguments is nonintegral.

4.3. Evaluation of C_4. The last example discussed here is

$$C_4 = \frac{1}{6} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\, dy\, dz\, dw}{xyzw\, (x + 1/x + y + 1/y + z + 1/z + w + 1/w)^2}.$$

To minimize the number of sums and to maximize the number of brackets this is rewritten as

$$\frac{1}{6} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{1+\epsilon} y^{1+\epsilon} z^{1+\epsilon} w^{1+\epsilon}\, dx\, dy\, dz\, dw}{[Axyzw(x+y) + zw(x+y) + xyzw(z+w) + xy(z+w)]^2}$$

with the intent of letting $\epsilon \to 0$ and $A \to 1$. As in the case of $C_{3,k}$, the *regulator* parameter ϵ is introduced to cure the divergence of the resulting expressions. Similarly, the parameter A is employed to divide the resulting sums into convergence groups according to the heuristic Rule 2.3. The denominator expands as

$$\sum_{\{n\}} \phi_{\{n\}}\, A^{n_1} x^{n_1+n_3+n_4} y^{n_1+n_3+n_4} z^{n_1+n_2+n_3} w^{n_1+n_2+n_3}$$

$$\times\, (x+y)^{n_1+n_2} (z+w)^{n_3+n_4} \langle 2 + n_1 + n_2 + n_3 + n_4 \rangle.$$

As before,

$$(x+y)^{n_1+n_2} = \sum_{n_5,n_6} \phi_{n_5,n_6}\, x^{n_5} y^{n_6} \frac{\langle -n_1 - n_2 + n_5 + n_6 \rangle}{\Gamma(-n_1 - n_2)}$$

and

$$(z+w)^{n_3+n_4} = \sum_{n_7,n_8} \phi_{n_7,n_8}\, z^{n_7} w^{n_8} \frac{\langle -n_3 - n_4 + n_7 + n_8 \rangle}{\Gamma(-n_3 - n_4)}.$$

These expansions of the integrand yield the bracket series

(4.29)
$$\frac{1}{6} \sum_{\{n\}} \phi_{\{n\}}\, A^{n_1} \langle 2 + n_1 + n_2 + n_3 + n_4 \rangle$$

$$\times\, \frac{\langle -n_1 - n_2 + n_5 + n_6 \rangle}{\Gamma(-n_1 - n_2)} \frac{\langle -n_3 - n_4 + n_7 + n_8 \rangle}{\Gamma(-n_3 - n_4)}$$

$$\times\, \langle 2 + \epsilon + n_1 + n_3 + n_4 + n_5 \rangle \langle 2 + \epsilon + n_1 + n_3 + n_4 + n_6 \rangle$$

$$\times\, \langle 2 + \epsilon + n_1 + n_2 + n_3 + n_7 \rangle \langle 2 + \epsilon + n_1 + n_2 + n_3 + n_8 \rangle.$$

The evaluation of this bracket series by Rules 2.2 and 2.3 yields hypergeometric series with arguments A (n_1, n_2, n_5, or n_6 chosen as the free index) and $1/A$ (n_3, n_4, n_7, or n_8 chosen as the free index). Either combination produces an expression for the integral C_4. Taking those with argument A (the indices n_5 and n_6 yield the same series; however, it is only taken into account once) gives

(4.30)
$$\frac{1}{12} A^{-\epsilon} \Gamma^2(\epsilon) \Gamma^2(1-\epsilon) \left(\frac{A^\epsilon}{1+2\epsilon}\, {}_2F_1\left(\begin{matrix} \frac{1}{2}+\epsilon, 1 \\ \frac{3}{2}+\epsilon \end{matrix} \middle| A \right) \right.$$

$$\left. + \frac{A^{-\epsilon}}{1-2\epsilon}\, {}_2F_1\left(\begin{matrix} \frac{1}{2}-\epsilon, 1 \\ \frac{3}{2}-\epsilon \end{matrix} \middle| A \right) - 2\, {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2} \end{matrix} \middle| A \right) \right).$$

As $\epsilon \to 0$, the limiting value is

$$(4.31) \qquad \frac{1}{24} \ln^2 A \ln\left(\frac{1+\sqrt{A}}{1-\sqrt{A}}\right) + \frac{1}{3\sqrt{A}} \left[\text{Li}_3(\sqrt{A}) - \text{Li}_3(-\sqrt{A})\right]$$

$$- \frac{\ln A}{6\sqrt{A}} \left[\text{Li}_2(\sqrt{A}) - \text{Li}_2(-\sqrt{A})\right].$$

Finally, the value of C_4 is obtained by taking $A \to 1$:

$$(4.32) \qquad C_4 = \frac{1}{3}\left[\text{Li}_3(1) - \text{Li}_3(-1)\right] = \frac{7}{12}\zeta(3).$$

This agrees with (4.3).

5. Analytic continuation of hypergeometric functions

The hypergeometric function $_pF_q$, defined by the series

$$(5.1) \qquad _pF_q(x) = {_pF_q}\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array}\bigg| x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

converges for all $x \in \mathbb{C}$ if $p < q+1$ and for $|x| < 1$ if $p = q+1$. In the remaining case, $p > q+1$, the series diverges for $x \neq 0$. The analytic continuation of the series $_{q+1}F_q$ has been recently considered in [**11, 12**]. In this section a brackets representation of the hypergeometric series is obtained and then employed to produce its analytic extension.

THEOREM 5.1. *The bracket representation of the hypergeometric function is given by*

$$_pF_q(x) = \sum_{\substack{n \\ t_1, \ldots, t_p \\ s_1, \ldots, s_q}} \phi_{n,\{t\},\{s\}} \; \left[(-1)^{q-1}x\right]^n \prod_{j=1}^{p} \frac{\langle a_j + n + t_j \rangle}{\Gamma(a_j)} \prod_{k=1}^{q} \frac{\langle 1 - b_k - n + s_k \rangle}{\Gamma(1 - b_k)}.$$

PROOF. This follows from (5.1) and the representations

$$(5.2) \qquad (a_j)_n = \frac{\Gamma(a_j + n)}{\Gamma(a_j)} = \frac{1}{\Gamma(a_j)} \int_0^{\infty} \tau^{a_j+n-1} e^{-\tau} \, d\tau = \sum_{t_j} \phi_{t_j} \frac{\langle a_j + n + t_j \rangle}{\Gamma(a_j)}$$

as well as

$$(5.3) \qquad \frac{1}{(b_k)_n} = (-1)^n \frac{\Gamma(1 - b_k - n)}{\Gamma(1 - b_k)} = (-1)^n \sum_{s_k} \phi_{s_k} \frac{\langle 1 - b_k - n + s_k \rangle}{\Gamma(1 - b_k)}$$

for the Pochhammer symbol. □

The bracket expression for the hypergeometric function given in Theorem 5.1 contains $p + q$ brackets and $p + q + 1$ indices (n, t_j and s_k). This leads to a full rank system

$$(5.4) \qquad \begin{aligned} a_j + n + t_j &= 0 & \text{for } 1 \leq j \leq p \\ 1 - b_k - n + s_k &= 0 & \text{for } 1 \leq k \leq q. \end{aligned}$$

of linear equations of size $(p+q+1) \times (p+q)$ and determinant 1. For each choice of an index as a free variable the method of brackets yields a one-dimensional series for the integral.

Series with n as a free variable. Solving (5.4) yields $t_j^* = -a_j - n$ and $s_k^* = -(1 - b_k) + n$ with $1 \le j \le p$ and $1 \le k \le q$. Rule 2.2 yields

$$\sum_{n=0}^{\infty} \frac{[(-1)^q x]^n}{n!} \prod_{j=1}^{p} \frac{\Gamma(n + a_j)}{\Gamma(a_j)} \prod_{k=1}^{q} \frac{\Gamma(-n + 1 - b_k)}{\Gamma(1 - b_k)} = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

This is the original series representation (5.1) of the hypergeometric function. In particular, in the case $q = p - 1$, this series converges for $|x| < 1$.

Series with t_i as a free variable. Fix an index i in the range $1 \le i \le p$ and solve (5.4) to get $n^* = -a_i - t_i$, as well as $t_j^* = t_i - a_j + a_i$ for $1 \le j \le p$, $j \ne i$, and $s_k^* = -(1 - b_k) - a_i - t_i$ for $1 \le k \le q$. The method of brackets then produces the series

$$\sum_{t_i} \phi_{t_i} \left[(-1)^{q-1} x \right]^{-t_i - a_i} \frac{\Gamma(t_i + a_i)}{\Gamma(a_i)} \prod_{j \ne i} \frac{\Gamma(a_j - a_i - t_i)}{\Gamma(a_j)} \prod_k \frac{\Gamma(1 - b_k + a_i + t_i)}{\Gamma(1 - b_k)}$$

which may be rewritten as

$$(5.5) \qquad (-x)^{-a_i} \prod_{j \ne i} \frac{\Gamma(a_j - a_i)}{\Gamma(a_j)} \prod_k \frac{\Gamma(b_k)}{\Gamma(b_k - a_i)}$$

$$\times \; {}_{q+1}F_{p-1} \left(\begin{array}{c} a_i, \{1 - b_k + a_i\}_{1 \le k \le q} \\ \{1 - a_j + a_i\}_{1 \le j \le p, j \ne i} \end{array} \middle| \frac{(-1)^{p+q-1}}{x} \right).$$

Recall that the initial hypergeometric series ${}_pF_q(x)$ converges for some $x \ne 0$ if and only if $p \le q + 1$. Hence, assuming that $p \le q + 1$, observe that the hypergeometric series (5.5) converges for some x if and only if $p = q + 1$.

Series with s_i as a free variable. Proceeding as in the previous case and choosing i in the range $1 \le i \le q$ and then s_i as the free index, gives

$$(5.6) \qquad \left[(-1)^{p+q-1} x \right]^{1 - b_i} \frac{\Gamma(b_i - 1)}{\Gamma(1 - b_i)} \prod_j \frac{\Gamma(1 - a_j)}{\Gamma(b_i - a_j)} \prod_{k \ne i} \frac{\Gamma(b_i - b_k)}{\Gamma(1 - b_k)}$$

$$\times \; {}_pF_q \left(\begin{array}{c} \{a_j + 1 - b_i\}_{1 \le j \le p} \\ 2 - b_i, \{1 - b_k + b_i\}_{1 \le k \le q, k \ne i} \end{array} \middle| x \right).$$

Summary. Assume $p = q + 1$ and sum up the series coming from the method of brackets converging in the common region $|x| > 1$. Rule 2.3 gives the analytic continuation

$$(5.7) \qquad {}_{q+1}F_q(x) = \sum_{i=1}^{q+1} (-x)^{-a_i} \prod_{j \ne i} \frac{\Gamma(a_j - a_i)}{\Gamma(a_j)} \prod_k \frac{\Gamma(b_k)}{\Gamma(b_k - a_i)}$$

$$\times \; {}_{q+1}F_q \left(\begin{array}{c} a_i, \{1 - b_k + a_i\}_{1 \le k \le q} \\ \{1 - a_j + a_i\}_{1 \le j \le q+1, j \ne i} \end{array} \middle| \frac{1}{x} \right)$$

for the series (5.1).

On the other hand, the $q+1$ functions coming from choosing n or s_i, $1 \le i \le q$, as the free variables form linearly independent solutions to the hypergeometric differential equation

$$(5.8) \qquad \prod_{j=1}^{q+1} \left(x \frac{d}{dx} + a_j \right) y = \prod_{k=1}^{q} \left(x \frac{d}{dx} + b_k \right) y$$

in a neighborhood of $x = 0$. Likewise, the $q+1$ functions (5.5) coming from choosing t_i, $1 \le i \le q + 1$, as the free variables form linearly independent solutions to (5.8) in a neighborhood of $x = \infty$.

EXAMPLE 5.2. For instance, if $p = 2$ and $q = 1$ then

$$(5.9) \qquad {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = (-x)^{-a} \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} {}_2F_1 \left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix} \middle| \frac{1}{x} \right)$$
$$+ (-x)^{-b} \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} {}_2F_1 \left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix} \middle| \frac{1}{x} \right).$$

This is entry 9.132.1 of [**7**]. On the other hand, the two functions

$$(5.10) \qquad {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right), \qquad x^{1-c} {}_2F_1 \left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix} \middle| x \right)$$

form a basis of the solutions to the second-order hypergeometric differential equation

$$(5.11) \qquad \left(x \frac{d}{dx} + a \right) \left(x \frac{d}{dx} + b \right) y = \left(x \frac{d}{dx} + c \right) y$$

in a neighborhood of $x = 0$.

6. Feynman diagram application

In Quantum Field Theory the permanent contrast between experimental measurements and theoretical models has been possible due to the development of novel and powerful analytical and numerical techniques in perturbative calculations. The fundamental problem that arises in perturbation theory is the actual calculation of the loop integrals associated to the Feynman diagrams, whose solution is specially difficult since these integrals contain in general both ultraviolet (UV) and infrared (IR) divergences. Using the dimensional regularization scheme, which extends the dimensionality of space-time by adding a fractional piece ($D = 4 - 2\epsilon$), it is possible to know the behavior of such divergences in terms of Laurent expansions with respect to the dimensional regulator ϵ when it tends to zero

As an illustration of the use of method of brackets, the Feynman diagram

(6.1)
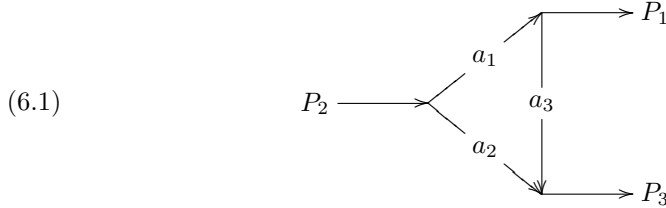
considered in [**2**] is resolved. In this diagram the propagator (or internal line) associated to the index a_1 has mass m and the other parameters are $P_1^2 = P_3^2 = 0$ and $P_2^2 = (P_1 + P_3)^2 = s$. The D-dimensional representation in Minkowski space is given by

$$(6.2) \qquad G = \int \frac{d^D q}{i\pi^{D/2}} \frac{1}{[(P_1 + q)^2 - m^2]^{a_1} [(P_3 - q)^2]^{a_2} [q^2]^{a_3}}.$$

In order to evaluate this integral, the Schwinger parametrization of (6.2) is considered (see [**8**] for details). This is given by

$$(6.3) \qquad G = \frac{(-1)^{-D/2}}{\prod_{j=1}^{3} \Gamma(a_j)} H$$

with H defined by

$$(6.4) \quad H = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\exp\left(x_1 m^2\right) \exp\left(-\frac{x_1 x_2}{x_1+x_2+x_3} s\right)}{(x_1+x_2+x_3)^{D/2}} \, dx_1 dx_2 dx_3.$$

To apply the method of brackets the exponential terms are expanded as

$$\exp\left(x_1 m^2\right) \exp\left(-\frac{x_1 x_2}{x_1+x_2+x_3} s\right) = \sum_{n_1,n_2} \phi_{n_1,n_2} (-1)^{n_1} m^{2n_1} s^{n_2} \frac{x_1^{n_1+n_2} x_2^{n_2}}{(x_1+x_2+x_3)^{n_2}},$$

and then (6.4) is transformed into

$$(6.5) \qquad \sum_{n_1,n_2} \phi_{n_1,n_2} (-m^2)^{n_1} s^{n_2} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \frac{x_1^{a_1+n_1+n_2-1} x_2^{a_2+n_2-1} x_3^{a_3-1}}{(x_1+x_2+x_3)^{D/2+n_2}} \, dx_1 dx_2 dx_3.$$

Further expanding

$$\frac{1}{(x_1+x_2+x_3)^{D/2+n_2}} = \sum_{n_3,n_4,n_5} \phi_{n_3,n_4,n_5} x_1^{n_3} x_2^{n_4} x_3^{n_5} \frac{\left\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \right\rangle}{\Gamma(\frac{D}{2} + n_2)},$$

and replacing into (6.5) and substituting the resulting integrals by the corresponding brackets yields

$$(6.6) \qquad H = \sum_{\{n\}} \phi_{\{n\}} (-1)^{n_1} m^{2n_1} s^{n_2} \frac{\left\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \right\rangle}{\Gamma(\frac{D}{2} + n_2)}$$
$$\times \ \langle a_1 + n_1 + n_2 + n_3 \rangle \langle a_2 + n_2 + n_4 \rangle \langle a_3 + n_5 \rangle.$$

This bracket series is now evaluated employing Rules 2.2 and 2.3. Possible choices for free variables are n_1, n_2, and n_4. The series associated to n_2 converges for $\left|\frac{s}{m^2}\right| < 1$, whereas the series associated to n_1, n_4 converge for $\left|\frac{m^2}{s}\right| < 1$. The following two representations for G follow from here.

THEOREM 6.1. *In the region* $\left|\frac{s}{m^2}\right| < 1$,

$$(6.7) \qquad H = \eta_2 \cdot {}_2F_1 \left(\begin{array}{c} a_1 + a_2 + a_3 - \frac{D}{2}, a_2 \\ \frac{D}{2} \end{array} \middle| \frac{s}{m^2} \right)$$

with η_2 *defined by*

$$\eta_2 = \left(-m^2\right)^{\frac{D}{2} - a_1 - a_2 - a_3} \frac{\Gamma(a_2) \Gamma(a_3) \Gamma\left(a_1 + a_2 + a_3 - \frac{D}{2}\right) \Gamma\left(\frac{D}{2} - a_2 - a_3\right)}{\Gamma\left(\frac{D}{2}\right)}.$$

THEOREM 6.2. *In the region* $|\frac{m^2}{s}| < 1$,

$$(6.8) \qquad H = \eta_1 \cdot {}_2F_1 \left(\begin{matrix} a_1 + a_2 + a_3 - \frac{D}{2}, 1 + a_1 + a_2 + a_3 - D \\ 1 + a_1 + a_3 - \frac{D}{2} \end{matrix} \middle| \frac{m^2}{s} \right)$$

$$+ \eta_4 \cdot {}_2F_1 \left(\begin{matrix} 1 + a_2 - \frac{D}{2}, a_2 \\ 1 - a_1 - a_3 + \frac{D}{2} \end{matrix} \middle| \frac{m^2}{s} \right)$$

with η_1, η_4 *defined by*

$$\eta_1 = s^{\frac{D}{2} - a_1 - a_2 - a_3} \frac{\Gamma(a_3)\Gamma\left(a_1 + a_2 + a_3 - \frac{D}{2}\right)\Gamma\left(\frac{D}{2} - a_1 - a_3\right)\Gamma\left(\frac{D}{2} - a_2 - a_3\right)}{\Gamma(D - a_1 - a_2 - a_3)},$$

$$\eta_4 = s^{-a_2}\left(-m^2\right)^{\frac{D}{2} - a_1 - a_3} \frac{\Gamma(a_2)\Gamma(a_3)\Gamma\left(a_1 + a_3 - \frac{D}{2}\right)\Gamma\left(\frac{D}{2} - a_2 - a_3\right)}{\Gamma\left(\frac{D}{2} - a_2\right)}.$$

These two solutions are now specialized to $a_1 = a_2 = a_3 = 1$. This situation is specially relevant, since when an arbitrary Feynman diagram is computed, the indices associated to the propagators are normally 1. Then, with $D = 4 - 2\epsilon$, the equations (6.7) and (6.8) take the form

$$(6.9) \qquad H = (-m^2)^{-1-\epsilon}\Gamma(\epsilon - 1){}_2F_1 \left(\begin{matrix} 1 + \epsilon, 1 \\ 2 - \epsilon \end{matrix} \middle| \frac{s}{m^2} \right)$$

for $|\frac{s}{m^2}| < 1$, as well as

$$(6.10) \quad H = s^{-1-\epsilon} \frac{\Gamma(-\epsilon)^2\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)} \left(1 - \frac{m^2}{s}\right)^{-2\epsilon} - m^{-2\epsilon} \frac{\Gamma(\epsilon)}{\epsilon s} {}_2F_1 \left(\begin{matrix} \epsilon, 1 \\ 1 - \epsilon \end{matrix} \middle| \frac{m^2}{s} \right)$$

for $|\frac{m^2}{s}| < 1$. Observe that these representations both have a pole at $\epsilon = 0$ of first order (for the second representation, each of the summands has a pole of second order which cancel each other).

7. Conclusions and future work

The method of brackets provides a very effective procedure to evaluate definite integrals over the interval $[0, \infty)$. The method is based on a heuristic list of rules on the bracket series associated to such integrals. In particular, a variety of examples that illustrate the power of this method has been provided. A rigorous validation of these rules as well as a systematic study of integrals from Feynman diagrams is in progress.

Acknowledgments

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