Core partitions into distinct parts and an analog of Euler's theorem

Armin Straub*
Department of Mathematics and Statistics
University of South Alabama

January 26, 2016

Abstract

A special case of an elegant result due to Anderson proves that the number of $(s, s+1)$-core partitions is finite and is given by the Catalan number C_s. Amdeberhan recently conjectured that the number of $(s, s+1)$-core partitions into distinct parts equals the Fibonacci number F_{s+1}. We prove this conjecture by enumerating, more generally, $(s, ds-1)$-core partitions into distinct parts. We do this by relating them to certain tuples of nested twin-free sets.

As a by-product of our results, we obtain a bijection between partitions into distinct parts and partitions into odd parts, which preserves the perimeter (that is, the largest part plus the number of parts minus 1). This simple but curious analog of Euler's theorem appears to be missing from the literature on partitions.

1 Introduction

A *partition* λ of n (for very good introductions see [And76] and [AE04]) is a finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$. The integers $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are referred to as the *parts* of λ, with λ_1 being the largest part and ℓ the number of parts. Such a partition λ is frequently represented by its *Young diagram*, which we take to be a left-justified array of square cells with ℓ rows such that the ith row consists of λ_i cells. To each cell u is assigned a *hook*, which is composed of the cell u itself as well as all cells to the right of u and below u. The *hook length* of u is the number of cells the hook consists of. A partition λ is said to be t-*core* if λ has no cell of hook length equal to t. An explanation of this terminology is given, for instance, in [AHJ14]. More generally, λ is said to be $(t_1, t_2, \ldots, t_r)$-core if λ is t-core for $t = t_1, t_2, \ldots, t_r$.

* *Email:* straub@southalabama.edu

The motivation to count partitions that are t-core for different values of t has been sparked by the following elegant result due to Anderson [And02].

Theorem 1.1. *The number of (s,t)-core partitions is finite if and only if s and t are coprime. In that case, this number is*

$$\frac{1}{s+t}\binom{s+t}{s}.$$

In particular, the number of $(s,s+1)$-core partitions is the Catalan number

$$C_s = \frac{1}{s+1}\binom{2s}{s} = \frac{1}{2s+1}\binom{2s+1}{s},$$

which also counts the number of Dyck paths of order s. Generalizations to $(s,s+1,\ldots,s+p)$-core partitions, including a relation to generalized Dyck paths, are given in [AL15].

In a different direction, Ford, Mai and Sze [FMS09] show that the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor s/2\rfloor + \lfloor t/2\rfloor}{\lfloor s/2\rfloor},$$

provided that s and t are coprime. More generally, Amdeberhan [Amd15] raises the interesting problem of counting the number of special partitions which are t-core for certain values of t. In particular, he conjectures the following count.

Conjecture 1.2. *The number of $(s,s+1)$-core partitions into distinct parts equals the Fibonacci number F_{s+1}.*

It is further conjectured in [Amd15] that the largest possible size of an $(s,s+1)$-core partition into distinct parts is $\lfloor s(s+1)/6\rfloor$, and that there is a unique such largest partition unless $s\equiv 1$ modulo 3, in which case there are two partitions of maximum size. Amdeberhan also provides a conjecture for the average size of these partitions. We do not pursue these more intricate, but very interesting, questions here (the interested reader is referred to, for instance, [OS07], [AHJ14], [SZ15], [CHW16], [Joh15], and the references therein, for the case of general core partitions, and [Xio15] for the case of $(s,s+1)$-core partitions into distinct parts). Instead, we focus on the most basic question on core partitions into distinct parts, namely to enumerate them. Ultimately, our main result is the following enumeration of (s,t)-core partitions into distinct parts for a two-parameter family of values (s,t).

Theorem 1.3. *Let $d,s\geq 1$. The number $N_d(s)$ of $(s,ds-1)$-core partitions into distinct parts is characterized by $N_d(1)=1$, $N_d(2)=d$ and, for $s\geq 3$,*

$$N_d(s) = N_d(s-1) + dN_d(s-2).$$

In particular, the case $d = 1$ clearly settles Conjecture 1.2. This special case has since been also independently proved by Xiong [Xio15]. Before giving a proof of Theorem 1.3 in Section 4, we discuss an elementary bijective proof of the special case $d = 1$ in Sections 2 and 3.

We do so, because a natural extension of our approach leads to a simple but curious analog of Euler's theorem on partitions into distinct (respectively odd) parts, which appears to be missing from the literature on partitions. Namely, we obtain a bijection between partitions into distinct parts on the one hand and partitions into odd parts on the other hand, which preserves the perimeter of the partitions. Here, following Corteel and Lovejoy [CL04, Section 4.2] (up to a shift by 1), we refer to the *perimeter* of a partition as the maximum part plus the number of parts minus 1 (equivalently, the perimeter of λ is the maximum hook length in λ).

Theorem 1.4. *The number of partitions into distinct parts with perimeter M is equal to the number of partitions into odd parts with perimeter M. Both are enumerated by the Fibonacci number F_M.*

Example 1.5. The partitions into distinct parts with perimeter 5 are (5), $(4,1)$, $(4,2)$, $(4,3)$ and $(3,2,1)$. The partitions into odd parts with perimeter 5 are (5), $(3,3,3)$, $(3,3,1)$, $(3,1,1)$ and $(1,1,1,1,1)$. In each case, there are $F_5 = 5$ many of these partitions.

While it appears natural, we have been unable to find the result in Theorem 1.4 in the existing literature. On the other hand, an intriguingly similar result of Euler is widely known: the number $D(n)$ of partitions of n into distinct parts equals the number $O(n)$ of partitions of n into odd parts. In other words, there is a bijection between partitions into distinct and odd parts, which preserves the size of the partitions. While there are bijective proofs (see, for instance, [AE04, Chapter 2.3]), Euler famously proved his claim using a very elegant manipulation of generating functions (see, for instance, [And76, Cor. 1.2] or [AE04, Chapter 5.2]). Namely, he observed that

$$\begin{aligned}\sum_{n\geq 0} D(n)x^n &= (1+x)(1+x^2)(1+x^3)\cdots \\ &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots = \sum_{n\geq 0} O(n)x^n.\end{aligned}$$

Several refinements of Euler's theorem due to Sylvester, Fine and Bousquet-Mélou–Eriksson are beautifully presented, for instance, in the book [AE04, Chapter 9] by Andrews and Eriksson.

Example 1.6. Bousquet-Mélou and Eriksson [BME97a], [BME97b] show that the number of lecture hall partitions of n with length k (these are special partitions of n into distinct parts) is equal to the number of partitions of n into odd parts with each part at most $2k-1$. Among other refinements, they also prove that the number of partitions of n into distinct parts with sign-alternating sum k is equal to the number of partitions of n into k odd parts. A corresponding combinatorial bijection is given in [KY99].

Example 1.7. Another refinement, found in [Fin88, (23.91)], shows that the number of partitions of n into distinct parts with maximum part M is equal to the number of partitions of n into odd parts such that the maximum part plus twice the number of parts is $2M+1$.

Example 1.8. The *rank* of a partition is the difference between the largest part and the number of parts. Note that the rank of a partition into distinct parts is always nonnegative. Then, we have [Fin88, (24.6)] that the number of partitions of n into odd parts with maximum part equal to $2M+1$ is equal to the number of partitions of n into distinct parts with rank $2M$ or $2M+1$.

A natural question is whether similarly interesting refinements exist for Theorem 1.4, that is, for partitions into distinct (respectively odd) parts with perimeter M.

2 Partitions with bounded hook lengths

We begin by proving the case $d = 1$ of Theorem 1.3, thus establishing Conjecture 1.2. The proof for the general case is then given in Section 4. Let F_s denote the Fibonacci numbers with $F_0 = 0$ and $F_1 = 1$.

Theorem 2.1. *There are F_{s+1} many $(s, s+1)$-core partitions into distinct parts.*

In preparation for Theorem 2.1, we first prove the following claim.

Lemma 2.2. *A partition into distinct parts is $(s, s+1)$-core if and only if it has perimeter strictly less than s.*

Proof. Recall that the perimeter of a partition λ is the maximum hook length in λ. We therefore need to show that, if λ is $(s, s+1)$-core, then λ is $(s, s+1, s+2, \ldots)$-core. Suppose otherwise, and let t be the smallest hook length in λ larger than s. By construction, λ is $(t-1, t-2)$-core. Consider the Young diagram of λ, and focus on a cell u with hook length t. A moment of reflection reveals that, since λ has distinct parts, the cell to the right of u has hook length $t-1$ or $t-2$. This contradicts the fact that λ is $(t-1, t-2)$-core, and so our claim must be true. $\square$

Proof of Theorem 2.1. By virtue of Lemma 2.2, we need to show that there are F_{s+1} many partitions into distinct parts with perimeter strictly less than s. One checks directly that this is true for $s = 1$ (in which case, $F_2 = 1$ and the relevant set of partitions consists of the empty partition only) and $s = 2$ (in which case, $F_3 = 2$ and the relevant set of partitions consists of the partition (1) and the empty partition).

Let $s \geq 3$ and, for the purpose of induction, suppose that the number of partitions into distinct parts with perimeter less than r is given by F_{r+1} for all $r < s$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition into distinct parts with perimeter less than s. Then exactly one of the following two cases applies:

(a) The largest part λ_1 satisfies $\lambda_1 > \lambda_2 + 1$. Then, consider the partition $\lambda' = (\lambda_1 - 1, \lambda_2, \lambda_3, \ldots)$. By the assumption on λ_1, the partition λ' still has distinct parts. On the other hand, the perimeter of λ' is one less than the perimeter of λ. In fact, λ' can be any of the F_s many partitions into distinct parts with perimeter less than $s - 1$.

(b) The largest part λ_1 satisfies $\lambda_1 = \lambda_2 + 1$. In that case, consider the partition $\lambda' = (\lambda_2, \lambda_3, \ldots)$, which has distinct parts. Clearly, the perimeter of λ' is two less than the perimeter of λ. Again, λ' can be any of the F_{s-1} many partitions into distinct parts with perimeter less than $s - 2$.

Taken together, we find that the number of partitions into distinct parts with perimeter less than s is given by $F_s + F_{s-1} = F_{s+1}$, as claimed. □

Now, we establish Theorem 1.4 via a natural variation of our proof of Theorem 2.1. Recall that the perimeter of a partition is the maximum hook length in the partition. In the spirit of Euler's result, Theorem 1.4 claims that, for $M \geq 1$, the number of partitions into distinct parts with perimeter M is equal to the number of partitions into odd parts with perimeter M, and that this common number is F_M.

Proof of Theorem 1.4. By Lemma 2.2, a partition into distinct parts is $(s, s+1)$-core if and only if it has perimeter at most $s - 1$. Hence, Theorem 2.1 can be rephrased as saying that there are F_{M+2} many partitions into distinct parts with perimeter at most M. Consequently, there are $F_M = F_{M+2} - F_{M+1}$ many partitions into distinct parts with perimeter exactly M. This verifies the first part of Theorem 1.4.

It remains to prove that there are also F_M partitions into odd parts with perimeter M. We proceed using a variation of our proof of Theorem 2.1. Again, it is straightforward to verify the claim for $M = 1$ and $M = 2$. For the purpose of induction suppose that, for all $m < M$, there are F_m many partitions into odd parts with perimeter m. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition into odd parts with perimeter M. Then exactly one of the following two cases applies:

(a) The largest part λ_1 satisfies $\lambda_1 > \lambda_2 + 1$. Then, consider the partition $\lambda' = (\lambda_1 - 2, \lambda_2, \lambda_3, \ldots)$. Clearly, the parts of λ' are all odd, and the perimeter of λ' is $M - 2$. Evidently, λ' can be any of the F_{M-2} many partitions into odd parts with perimeter $M - 2$.

(b) The largest part λ_1 satisfies $\lambda_1 = \lambda_2$. In that case, consider the partition $\lambda' = (\lambda_2, \lambda_3, \ldots)$. Clearly, the parts of λ' are all odd, and the perimeter of λ' is $M - 1$. Again, λ' can be any of the F_{M-1} many partitions into odd parts with perimeter $M - 1$.

Taken together, we find that the number of partitions into odd parts with perimeter M is given by $F_{M-2} + F_{M-1} = F_M$. $\square$

3 An explicit bijection

Partition theorists are often interested in bijective proofs of statements of equinumerosity. In the present discussion, a combination and comparison of the recursive proofs of Theorems 1.4 and 2.1 does yield an explicit bijection between partitions into distinct parts with perimeter M and partitions into odd parts with perimeter M.

Let $\mathcal{C}$ be the set of all compositions with parts 1 and 2, and such that the last part is not a 2. For instance, $\mathcal{C}$ contains the following compositions $\mu = (\mu_1, \mu_2, \ldots)$ of $|\mu| = \mu_1 + \mu_2 + \ldots = n$, for $n = 0, 1, \ldots 5$.

compositions μ in $\mathcal{C}$	$\|\mu\|$	#
$()$	0	1
(1)	1	1
$(1,1)$	2	1
$(1,1,1), (2,1)$	3	2
$(1,1,1,1), (1,2,1), (2,1,1)$	4	3

Table 1: Small compositions $\mu \in \mathcal{C}$ ranked by $|\mu|$

It is straightforward to enumerate compositions of M in $\mathcal{C}$.

Lemma 3.1. *For $M \geq 1$, there are F_M many compositions $\mu \in \mathcal{C}$ with $|\mu| = M$.*

For instance, a well-known equivalent version of this count appears as an exercise in [Sta97, Chapter 1, Exercise 14(c)], where the reader is asked to show that the number of compositions μ of M into parts 1 and 2 is F_{M+1}.

Next, we introduce bijections between $\mathcal{C}$ and the sets of partitions into distinct (respectively, odd) parts. Combining these two bijections, we then obtain a bijection, preserving perimeters, between partitions into distinct parts and partitions into odd parts. In particular, this fact implies Theorems 1.4 and 2.1.

Theorem 3.2. *The map $\mu \mapsto \lambda_d(\mu)$, described below, is a bijection between $\mathcal{C}$ and the set of partitions into distinct parts. Likewise, the map $\mu \mapsto \lambda_o(\mu)$ is a bijection between $\mathcal{C}$ and the set of partitions into odd parts. Moreover, the perimeter of the partitions $\lambda_d(\mu)$ and $\lambda_o(\mu)$ is $|\mu|$.*

Proof. Let $\mu \in \mathcal{C}$. We assign a partition $\lambda_d = \lambda_d(\mu)$ to μ by the following recursive recipe. If $\mu = ()$ or $\mu = (1)$, then $\lambda_d = \mu$. Otherwise, write $\mu = (\mu_1, \mu')$ with $\mu_1 \in \{1, 2\}$ and $\mu' \in \mathcal{C}$. Suppose that $\lambda' = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is the partition assigned to μ'.

(a) If $\mu_1 = 1$, then $\lambda_d = (\lambda_1 + 1, \lambda_2, \lambda_3, \ldots)$.

(b) If $\mu_1 = 2$, then $\lambda_d = (\lambda_1 + 1, \lambda_1, \lambda_2, \lambda_3, \ldots)$.

By construction, the partition λ_d has distinct parts. In fact, it is straightforward to verify (in the spirit of the proof of Theorem 2.1) that the map $\mu \mapsto \lambda_d(\mu)$ describes a bijection between $\mathcal{C}$ and the set of partitions into distinct parts.

Analogously, we assign a partition $\lambda_o = \lambda_o(\mu)$ to $\mu \in \mathcal{C}$ as follows. Again, if $\mu = ()$ or $\mu = (1)$, then $\lambda_o = \mu$. Otherwise, write $\mu = (\mu_1, \mu')$ with $\mu_1 \in \{1, 2\}$ and $\mu' \in \mathcal{C}$. Suppose that $\lambda' = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is the partition assigned to μ'.

(a) If $\mu_1 = 1$, then $\lambda_o = (\lambda_1, \lambda_1, \lambda_2, \lambda_3, \ldots)$.

(b) If $\mu_1 = 2$, then $\lambda_o = (\lambda_1 + 2, \lambda_2, \lambda_3, \ldots)$.

Then, the map $\mu \mapsto \lambda_o(\mu)$ describes a bijection between $\mathcal{C}$ and the set of partitions into odd parts.

Combining these two bijections, we have a bijection between partitions λ_d into distinct parts and partitions λ_o into odd parts.

$$\lambda_d = \lambda_d(\mu) \quad \leftrightarrow \quad \mu \quad \leftrightarrow \quad \lambda_o(\mu) = \lambda_o$$

It also follows from the respective constructions that the partitions $\lambda_d(\mu)$ and $\lambda_o(\mu)$ both have perimeter $|\mu|$. $\square$

Example 3.3. The following table lists all 13 partitions into distinct (respectively, odd) parts with perimeter at most 5, together with the composition μ in $\mathcal{C}$ they get matched with.

μ	λ_d	λ_o
$()$	$()$	$()$
(1)	(1)	(1)
$(1,1)$	(2)	$(1,1)$
$(1,1,1)$	(3)	$(1,1,1)$
$(2,1)$	$(2,1)$	(3)
$(1,1,1,1)$	(4)	$(1,1,1,1)$
$(1,2,1)$	$(3,1)$	$(3,3)$
$(2,1,1)$	$(3,2)$	$(3,1)$
$(1,1,1,1,1)$	(5)	$(1,1,1,1,1)$
$(1,1,2,1)$	$(4,1)$	$(3,3,3)$
$(1,2,1,1)$	$(4,2)$	$(3,3,1)$
$(2,1,1,1)$	$(4,3)$	$(3,1,1)$
$(2,2,1)$	$(3,2,1)$	(5)

In particular, note that the composition $\mu = (1, 2, 1)$ corresponds to the partitions

$$\begin{aligned} \lambda_d(\mu) &= (3,1), \\ \lambda_o(\mu) &= (3,3). \end{aligned}$$

This illustrates that, while the present bijection between partitions into distinct parts and partitions into odd parts preserves the perimeter, it does not preserve the size of the partitions. Therefore, our bijection is of a rather different nature compared to the bijections underlying Euler's theorem and its generalizations.

4 A generalization

Recall that we showed in Theorem 2.1 that, as conjectured by Amdeberhan [Amd15], $(s-1, s)$-core partitions into distinct parts are counted by the Fibonacci numbers F_s. In this section, we generalize this result and enumerate $(s, ds-1)$-core partitions into distinct parts for any $d \geq 1$. That is, we prove Theorem 1.3 from the introduction, which is restated here for the reader's convenience.

Theorem 4.1. *The number $N_d(s)$ of $(s, ds-1)$-core partitions into distinct parts is characterized by $N_d(1) = 1$, $N_d(2) = d$ and, for $s \geq 3$,*

$$N_d(s) = N_d(s-1) + dN_d(s-2). \tag{1}$$

We are confident that a suitable generalization of our proof of Theorem 2.1 can be used to prove this result. Since the details appear to be somewhat more technical, we instead offer an alternative proof, inspired by the approach taken in [Xio15].

Example 4.2. Versions of the numbers $N_d(s)$ in Theorem 4.1 have been studied in the literature since Lucas, and are usually referred to as generalized Fibonacci numbers or generalized Fibonacci polynomials (in the variable d). For further information and references, we refer the interested reader to the recent paper [ACMS14]. The first few polynomials $N_d(s)$, for $s = 1, 2, \ldots, 7$, are

$$1, \quad d, \quad 2d, \quad d(d+2), \quad d(3d+2), \quad d(d^2+5d+2), \quad d(4d^2+7d+2).$$

Of course, we recover the usual Fibonacci numbers upon setting $d = 1$.

In preparation for the proof of Theorem 4.1, we say that a set $X \subseteq \mathbb{Z}$ is *twin-free* if there is no $x \in X$ such that $\{x, x+1\} \subseteq X$. As the following result shows, the number of tuples of nested twin-free sets satisfies the same recursive relation that is claimed for the core partitions in Theorem 4.1. Note, however, that the initial conditions differ.

Lemma 4.3. *Let $M_d(s)$ denote the number of tuples $(X_1, X_2, \ldots, X_d)$ of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \ldots \subseteq X_1 \subseteq \{1, 2, \ldots, s-1\}$. Then, $M_d(1) = 1$, $M_d(2) = d+1$ and, for $s \geq 3$,*

$$M_d(s) = M_d(s-1) + dM_d(s-2). \tag{2}$$

Proof. Clearly, $M_d(1) = 1$ because in that case the only tuple $(X_1, X_2, \ldots, X_d)$ is the one with $X_1 = X_2 = \ldots = X_d = \{\}$. On the other hand, $M_d(2) = d+1$

because then all tuples $(X_1, X_2, \dots, X_d)$ are of the form $X_j = \{1\}$, if $j \le J$, and $X_j = \{\}$, if $j > J$, for some $J \in \{0, 1, \dots, d\}$.

We may therefore suppose that $s \ge 2$. Let $(X_1, X_2, \dots, X_d)$ be a tuple of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \dots \subseteq X_1 \subseteq \{1, 2, \dots, s-1\}$. Then, exactly one of the following two possibilities is true:

(a) None of the sets $X_1, X_2, \dots, X_d$ contains $s-1$.

(b) There is an index $J \in \{1, 2, \dots, d\}$ such that $s-1 \in X_j$ for all $j \le J$ and $s-1 \notin X_j$ for all $j > J$.

In case (a), our tuple $(X_1, X_2, \dots, X_d)$ is one of the $M_d(s-1)$ many tuples of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \dots \subseteq X_1 \subseteq \{1, 2, \dots, s-2\}$.

On the other hand, suppose case (b) holds with $J \in \{1, 2, \dots, d\}$. In that case, $s-1 \in X_1$. Since X_1 is twin-free it follows that $s-2 \notin X_1$, and hence $s-2$ is not contained in any of the sets $X_1, X_2, \dots, X_d$. Let $X'_1, X'_2, \dots, X'_d$ be the sets obtained from $X_1, X_2, \dots, X_d$ by removing $s-1$ from these sets. That is, $X'_j = X_j - \{s-1\}$. Observe that the tuple $(X'_1, X'_2, \dots, X'_d)$ can be any of the $M_d(s-2)$ many tuples of twin-free sets such that $X'_d \subseteq X'_{d-1} \subseteq \dots \subseteq X'_1 \subseteq \{1, 2, \dots, s-3\}$. Since $(X'_1, X'_2, \dots, X'_d)$ together with the value of J determines $(X_1, X_2, \dots, X_d)$, we conclude that case (b) accounts for exactly $dM_d(s-2)$ many tuples.

The recursive relation (2) follows upon combining these two cases. $\square$

Lemma 4.4. *$(s, ds-1)$-core partitions into distinct parts are in bijective correspondence with tuples $(X_1, X_2, \dots, X_d)$ of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \dots \subseteq X_1 \subseteq \{1, 2, \dots, s-1\}$ and $s-1 \notin X_d$.*

Proof. Following [Xio15], given a partition λ, we denote with $\beta(\lambda)$ the set of hook lengths $h(u)$ where u is a cell in the first column of λ. Clearly, the set $\beta(\lambda)$ uniquely determines λ. Moreover, λ is t-core if and only if, for any $x \in \beta(\lambda)$ with $x \ge t$, we always have $x - t \in \beta(\lambda)$ [Xio15, Lemma 2.1]. In particular, if λ is t-core, then λ is also nt-core for any $n \ge 1$. This implies that an $(s, ds-1)$-core partition into distinct parts is also $(ds-1, ds)$-core and hence, by Lemma 2.2, has perimeter (maximum hook length) at most $ds-2$.

Let λ be an $(s, ds-1)$-core partition into distinct parts. Equivalently, λ is an s-core partition into distinct parts with perimeter at most $ds-2$. Therefore, the set $\beta(\lambda)$ can be any twin-free set

$$\beta(\lambda) = \beta_1(\lambda) \cup \beta_2(\lambda) \cup \dots \cup \beta_{d-1}(\lambda) \cup \beta_d(\lambda)$$

where

$$\beta_j(\lambda) \subseteq \{(j-1)s+1, (j-1)s+2, \dots, js-1\},$$

for $j \in \{1, 2, \dots, d-1\}$, and

$$\beta_d(\lambda) \subseteq \{(d-1)s+1, (d-1)s+2, \dots, ds-2\}.$$

Attach to λ the tuple $(X_1, X_2, \ldots, X_d)$ with

$$X_j = \{x - (j-1)s \ : \ x \in \beta_j(\lambda)\}\,.$$

By construction, $X_j \subseteq \{1, 2, \ldots, s-1\}$. Since λ has distinct parts, the sets X_j are all twin-free. Recall that the condition that λ is s-core is equivalent to the following: if $x \in \beta_j(\lambda)$ with $j > 1$, then $x - s \in \beta_{j-1}(\lambda)$. This translates into $X_d \subseteq X_{d-1} \subseteq \ldots \subseteq X_1$. Finally, $s - 1 \notin X_d$ because $ds - 1 \notin \beta(\lambda)$. Since there are no further restrictions on the sets X_j, we have arrived at the bijective correspondence, as promised. □

Now, we are in a comfortable position to prove Theorem 4.1.

Proof of Theorem 4.1. In light of the bijective correspondence established in Lemma 4.4, $N_d(s)$ equals the number of tuples $(X_1, X_2, \ldots, X_d)$ of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \ldots \subseteq X_1 \subseteq \{1, 2, \ldots, s-1\}$ and $s - 1 \notin X_d$. We need to show that $N_d(1) = 1$, $N_d(2) = d$ and $N_d(s) = N_d(s-1) + dN_d(s-2)$. This is clearly a variation of Lemma 4.3 and, indeed, we can prove it along the same lines.

As in the proof of Lemma 4.3, we see that $N_d(1) = 1$, $N_d(2) = d$, the only difference being that in the latter case one tuple is excluded due to the condition $s-1 \notin X_d$. Therefore, consider the case $s \geq 2$. Let $(X_1, X_2, \ldots, X_d)$ be a tuple of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \ldots \subseteq X_1 \subseteq \{1, 2, \ldots, s-1\}$ and $s - 1 \notin X_d$. Then, exactly one of the following two possibilities is true:

(a) None of the sets $X_1, X_2, \ldots, X_d$ contains $s - 1$.

(b) There is an index $J \in \{1, 2, \ldots, d-1\}$ such that $s - 1 \in X_j$ for all $j \leq J$ and $s - 1 \notin X_j$ for all $j > J$.

In case (a), our tuple $(X_1, X_2, \ldots, X_d)$ is one of the $M_d(s-1)$ many tuples from Lemma 4.3 of twin-free sets such that $X_d \subseteq X_{d-1} \subseteq \ldots \subseteq X_1 \subseteq \{1, 2, \ldots, s-2\}$.

On the other hand, suppose case (b) holds with $J \in \{1, 2, \ldots, d-1\}$. As in the proof of Lemma 4.3, let $X'_j = X_j - \{s-1\}$. Again, the resulting tuple $(X'_1, X'_2, \ldots, X'_d)$ can be any of the $M_d(s-2)$ many tuples of twin-free sets such that $X'_d \subseteq X'_{d-1} \subseteq \ldots \subseteq X'_1 \subseteq \{1, 2, \ldots, s-3\}$. Since $(X'_1, X'_2, \ldots, X'_d)$ together with the value of J determines $(X_1, X_2, \ldots, X_d)$, we conclude that case (b) accounts for exactly $(d-1)M_d(s-2)$ many tuples.

Combining these two cases, we arrive at

$$N_d(s) = M_d(s-1) + (d-1)M_d(s-2). \tag{3}$$

The asserted recurrence relation (1) for $N_d(s)$ therefore follows from the recurrence relation (2) for $M_d(s)$. Indeed, for all $s \geq 5$,

$$\begin{aligned} N_d(s) &= [M_d(s-2) + dM_d(s-3)] + (d-1)[M_d(s-3) + dM_d(s-4)] \\ &= [M_d(s-2) + (d-1)M_d(s-3)] + d[M_d(s-3) + (d-1)M_d(s-4)] \\ &= N_d(s-1) + dN_d(s-2). \end{aligned}$$

It only remains to verify initial values. By (3), we have $N_d(3) = M_d(2) + (d-1)M_d(1) = 2d$ and $N_d(4) = M_d(3) + (d-1)M_d(2) = dM_d(2) + dM_d(1) = d^2 + 2d$. These values indeed also satisfy the recursive relation (1) for $N_d(s)$, since $N_d(2) + dN_d(1) = 2d$ and $N_d(3) + dN_d(2) = d^2 + 2d$. □

5 Conclusion

We proved an analog for partitions into distinct parts of Anderson's Theorem 1.1 specialized to $(s, s+1)$-core partitions. More generally, in Theorem 1.3, we enumerated $(s, ds-1)$-core partitions into distinct parts. It would be interesting to further generalize this result and determine a count for (s,t)-core partitions into distinct parts, for any coprime s and t. In this direction, we offer the following conjecture for further motivation.

Conjecture 5.1. *If s is odd, then the number of $(s, s+2)$-core partitions into distinct parts equals 2^{s-1}.*

This claim is based on experimental evidence and has been verified for $s < 20$ after listing all relevant partitions.

Example 5.2. For $s = 3$, the four $(3,5)$-core partitions into distinct parts are

$$\{\}, \quad \{1\}, \quad \{2\}, \quad \{3,1\}.$$

For $s = 5$, the sixteen $(5,7)$-core partitions into distinct parts are

$$\begin{gathered}\{\}, \quad \{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}, \quad \{2,1\}, \quad \{3,1\}, \quad \{5,1\}, \quad \{3,2\}, \quad \{4,2,1\}, \\ \{6,2,1\}, \quad \{4,3,1\}, \quad \{7,3,2\}, \quad \{5,4,2,1\}, \quad \{8,4,3,1\}, \quad \{9,5,4,2,1\}.\end{gathered}$$

Note that the largest occurring size among these partitions is $3 + 1 = 4$, for $s = 3$, and $9 + 5 + 4 + 2 + 1 = 21$, for $s = 5$. For $s = 3, 5, \ldots, 17$, the largest possible sizes of $(s, s+2)$-core partitions into distinct parts are

$$4, 21, 65, 155, 315, 574, 966, 1530.$$

Based on the initial data, it appears that there is a unique partition of this largest size, and that the largest possible size of an $(s, s+2)$-core partition into distinct parts is $\frac{1}{384}(s^2-1)(s+3)(5s+17)$. This partition of largest size appears to have both the highest number of parts (namely, $\frac{1}{8}(s-1)(s+5)$ many) and the largest part (namely, a part of size $\frac{3}{8}(s^2-1)$). After $\{3,1\}$ and $\{9,5,4,2,1\}$, the next such unique largest partitions are

$$\{18, 12, 11, 7, 6, 5, 3, 2, 1\}, \quad \{30, 22, 21, 15, 14, 13, 9, 8, 7, 6, 4, 3, 2, 1\}.$$

We hope that Conjecture 5.1 together with the results in this paper provide clues for enumerating (s,t)-core partitions into distinct parts. Table 2 lists the number of such partitions for $s, t \leq 12$. Observe, in particular, the occurrence of the Fibonacci numbers next to the main diagonal, in accordance with Theorem 2.1.

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
11	1	6	8	15	21	32	64	101	256	89	∞	144
12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

Table 2: The number of (s,t)-core partitions into distinct parts for $s,t \leq 12$

It would further be interesting, but appears to be harder, to enumerate (s,t)-core partitions into odd parts.

Acknowledgements. I thank Tewodros Amdeberhan for introducing me to (s,t)-core partitions and his conjecture, as well as for many interesting discussions, comments and suggestions. I am also grateful to George E. Andrews, Bruce C. Berndt, Robert Osburn and Wadim Zudilin for comments on an earlier version of this paper.

References

[Amd15] T. Amdeberhan. Theorems, problems and conjectures. *Preprint*, July 2015. Available at: http://arxiv.org/abs/1207.4045v6.

[ACMS14] T. Amdeberhan, X. Chen, V. H. Moll, and B. E. Sagan. Generalized Fibonacci polynomials and Fibonomial coefficients. *Annals of Combinatorics*, 18(4):541–562, October 2014.

[AL15] T. Amdeberhan and E. S. Leven. Multi-cores, posets, and lattice paths. *Advances in Applied Mathematics*, 71:1–13, October 2015.

[And02] J. Anderson. Partitions which are simultaneously t_1- and t_2-core. *Discrete Mathematics*, 248(1-3):237–243, April 2002.

[And76] G. E. Andrews. *The Theory of Partitions.* Addison-Wesley, 1976.

[AE04] G. E. Andrews and K. Eriksson. *Integer Partitions.* Cambridge University Press, 2004.

[AHJ14] D. Armstrong, C. R. H. Hanusa, and B. C. Jones. Results and conjectures on simultaneous core partitions. *European Journal of Combinatorics*, 41:205–220, October 2014.

[BME97a] M. Bousquet-Mélou and K. Eriksson. Lecture hall partitions. *The Ramanujan Journal*, 1(1):101–111, March 1997.

[BME97b] M. Bousquet-Mélou and K. Eriksson. Lecture hall partitions II. *The Ramanujan Journal*, 1(2):165–185, June 1997.

[CHW16] W. Chen, H. Huang, and L. Wang. Average size of a self-conjugate (s,t)-core partition. *Proceedings of the American Mathematical Society*, 144(4):1391–1399, 2016.

[CL04] S. Corteel and J. Lovejoy. Overpartitions. *Transactions of the American Mathematical Society*, 356(4):1623–1635, 2004.

[Fin88] N. J. Fine. *Basic hypergeometric series and applications*, volume 27 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[FMS09] B. Ford, H. Mai, and L. Sze. Self-conjugate simultaneous p- and q-core partitions and blocks of A_n. *Journal of Number Theory*, 129(4):858–865, April 2009.

[Joh15] P. Johnson. Lattice points and simultaneous core partitions. *Preprint*, February 2015. Available at: http://arxiv.org/abs/1502.07934.

[KY99] D. Kim and A. J. Yee. A note on partitions into distinct parts and odd parts. *The Ramanujan Journal*, 3(2):227–231, June 1999.

[OS07] J. B. Olsson and D. Stanton. Block inclusions and cores of partitions. *Aequationes mathematicae*, 74(1-2):90–110, September 2007.

[Sta97] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997.

[SZ15] R. P. Stanley and F. Zanello. The Catalan case of Armstrong's conjecture on simultaneous core partitions. *SIAM Journal on Discrete Mathematics*, 29(1):658–666, January 2015.

[Xio15] H. Xiong. Core partitions with distinct parts. *Preprint*, August 2015. Available at: http://arxiv.org/abs/1508.07918v1.