CERTAIN INTEGRALS ARISING FROM RAMANUJAN'S NOTEBOOKS

BRUCE C. BERNDT AND ARMIN STRAUB

Abstract. In his third notebook, Ramanujan claims that

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x + \frac{\pi}{2}\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x = 0.$$

In a following cryptic line, which only became visible in a recent reproduction of Ramanujan's notebooks, Ramanujan indicates that a similar relation existed if $\log x$ were replaced by $\log^2 x$ in the first integral and $\log x$ were inserted in the integrand of the second integral. One of the goals of the present paper is to prove this claim by contour integration. We further establish general theorems similarly relating large classes of infinite integrals and illustrate these by several examples.

1. Introduction

If you attempt to find the values of the integrals

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x \quad \text{and} \quad \int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x, \quad n>0, \tag{1.1}$$

by consulting tables such as those of Gradshteyn and Ryzhik [3] or by invoking a computer algebra system such as *Mathematica*, you will be disappointed, if you hoped to evaluate these integrals in closed form, that is, in terms of elementary functions. On the other hand, the latter integral above can be expressed in terms of the exponential integral $\mathrm{Ei}(x)$ [3, formula 3.723, no. 1]. Similarly, if $1/(x^2+1)$ is replaced by any even rational function with the degree of the denominator at least one greater than the degree of the numerator, it does not seem possible to evaluate any such integral in closed form.

However, in his third notebook, on page 391 in the pagination of the second volume of [5], Ramanujan claims that the two integrals in (1.1) are simple multiples of each other. More precisely,

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x + \frac{\pi}{2}\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x = 0. \tag{1.2}$$

Moreover, to the left of this entry, Ramanujan writes, "contour integration." We now might recall a couple sentences of G. H. Hardy from the introduction to Ramanujan's *Collected Papers* [4, p. xxx], "...he had [before arriving in England] never heard of ...Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was." On the following page, Hardy further wrote, "In a few years' time he had a very tolerable knowledge of the theory of functions ...." Generally, the entries in Ramanujan's notebooks were recorded by him in approximately the years

1904–1914, prior to his departure for England. However, there is evidence that some of the entries in his third notebook were recorded while he was in England. Indeed, in view of Hardy's remarks above, almost certainly, (1.2) is such an entry. A proof of (1.2) by contour integration was given by the first author in his book [2, pp. 329–330].

The identity (1.2) is interesting because it relates in a simple way two integrals that we are unable to individually evaluate in closed form. On the other hand, the simpler integrals

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\mathrm{d}x = \frac{\pi e^{-n}}{2} \quad \text{and} \quad \int_{-\infty}^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x = 0 \tag{1.3}$$

have well-known and trivial evaluations, respectively.

With the use of the most up-to-date photographic techniques, a new edition of *Ramanujan's Notebooks* [5] was published in 2012 to help celebrate the 125th anniversary of Ramanujan's birth. The new reproduction is vastly clearer and easier to read than the original edition. When the first author reexamined (1.2) in the new edition, he was surprised to see that Ramanujan made a further claim concerning (1.2) that was not visible in the original edition of [5]. In a cryptic one line, he indicated that a relation similar to (1.2) existed if $\log x$ were replaced by $\log^2 x$ in the first integral and $\log x$ were inserted in the integrand of the second integral of (1.2). One of the goals of the present paper is to prove (by contour integration) this unintelligible entry in the first edition of the notebooks [5]. Secondly, we establish general theorems relating large classes of infinite integrals for which individual evaluations in closed form are not possible by presently known methods. Several further examples are given.

2. Ramanujan's Extension of (1.2)

We prove the entry on page 391 of [5] that resurfaced with the new printing of [5].

Theorem 2.1. *We have*

$$\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x + \frac{2}{\pi}\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x \,\mathrm{d}x = 0 \tag{2.1}$$

and

$$\int_0^\infty \frac{\sin(nx)\log x}{x^2+1}\mathrm{d}x + \frac{1}{\pi}\int_0^\infty \frac{\cos(nx)\log^2 x}{x^2+1}\mathrm{d}x = \frac{\pi^2 e^{-n}}{8}. \tag{2.2}$$

Proof. Define a branch of $\log z$ by $-\frac{1}{2}\pi < \theta = \arg z \leq \frac{3}{2}\pi$. We integrate

$$f(z) := \frac{e^{inz}\log^2 z}{z^2+1}$$

over the positively oriented closed contour $C_{R,\varepsilon}$ consisting of the semi-circle C_R given by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $[-R, -\varepsilon]$, the semi-circle C_ε given by $z = \varepsilon e^{i\theta}$, $\pi \geq \theta \geq 0$, and $[\varepsilon, R]$, where $0 < \varepsilon < 1$ and $R > 1$. On the interior of $C_{R,\varepsilon}$ there is a simple pole at $z = i$, and so by the residue theorem,

$$\int_{C_{R,\varepsilon}} f(z)\mathrm{d}z = 2\pi i \frac{e^{-n}\cdot(-\frac{1}{4}\pi^2)}{2i} = -\frac{e^{-n}\pi^3}{4}. \tag{2.3}$$

Parameterizing the respective semi-circles, we can readily show that

$$\int_{C_\varepsilon} f(z)\mathrm{d}z = o(1), \tag{2.4}$$

as $\varepsilon \to 0$, and

$$\int_{C_R} f(z)\mathrm{d}z = o(1), \tag{2.5}$$

as $R \to \infty$. Hence, letting $\varepsilon \to 0$ and $R \to \infty$ and combining (2.3)–(2.5), we conclude that

$$\begin{aligned} -\frac{e^{-n}\pi^3}{4} &= \int_{-\infty}^{0} \frac{e^{inx}(\log|x| + i\pi)^2}{x^2+1}\mathrm{d}x + \int_0^\infty \frac{e^{inx}\log^2 x}{x^2+1}\mathrm{d}x \qquad (2.6)\\ &= \int_0^\infty \frac{(\cos(nx) - i\sin(nx))(\log x + i\pi)^2}{x^2+1}\mathrm{d}x + \int_0^\infty \frac{(\cos(nx) + i\sin(nx))\log^2 x}{x^2+1}\mathrm{d}x. \end{aligned}$$

If we equate real parts in (2.6), we find that

$$-\frac{e^{-n}\pi^3}{4} = \int_0^\infty \frac{\cos(nx)\{2\log^2 x - \pi^2\} + 2\pi\sin(nx)\log x}{x^2+1}\mathrm{d}x. \tag{2.7}$$

It is easy to show, e.g., by contour integration, that

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\mathrm{d}x = \frac{\pi e^{-n}}{2}. \tag{2.8}$$

(In his Quarterly Reports, Ramanujan derived (2.8) by a different method [1, p. 322].) Putting this evaluation in (2.7), we readily deduce (2.2). If we equate imaginary parts in (2.6), we deduce that

$$0 = \int_0^\infty \frac{\pi^2\sin(nx) + 2\pi\cos(nx)\log x}{x^2+1}\mathrm{d}x,$$

from which the identity (2.1) follows. □

3. A Second Approach to the Entry at the Top of Page 391

Theorem 3.1. *For $s \in (-1, 2)$ and $n \geq 0$,*

$$\frac{\pi}{2}e^{-n} = \int_0^\infty \frac{\cos(nx - \pi s/2)}{x^2+1}x^s\mathrm{d}x. \tag{3.1}$$

Before indicating a proof of Theorem 3.1, let us see how the integral (3.1) implies Ramanujan's integral relations (2.1) and (2.2). Essentially, all we have to do is to take derivatives of (3.1) with respect to s (and interchange the order of differentiation and integration); then, upon setting $s = 0$, we deduce (2.1) and (2.2).

First, note that upon setting $s = 0$ in (3.1), we obtain (2.8). On the other hand, taking a derivative of (3.1) with respect to s, and then setting $s = 0$, we find that

$$0 = \int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x + \frac{\pi}{2}\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x, \tag{3.2}$$

which is the formula (2.1) that Ramanujan recorded on page 391. Similarly, taking two derivatives of (3.1) and then putting $s = 0$, we arrive at

$$0 = \int_0^\infty \frac{\cos(nx)}{x^2+1} \log^2 x \, \mathrm{d}x + \pi \int_0^\infty \frac{\sin(nx)}{x^2+1} \log x \, \mathrm{d}x - \frac{\pi^2}{4} \int_0^\infty \frac{\cos(nx)}{x^2+1} \mathrm{d}x,$$

which, using (2.8), simplifies to

$$\frac{\pi^3}{8} e^{-n} = \int_0^\infty \frac{\cos(nx)}{x^2+1} \log^2 x \, \mathrm{d}x + \pi \int_0^\infty \frac{\sin(nx)}{x^2+1} \log x \, \mathrm{d}x. \tag{3.3}$$

Note that this is Ramanujan's previously unintelligible formula (2.2). If we likewise take m derivatives before setting $s = 0$, we obtain the following general set of relations connecting the integrals

$$I_m := \int_0^\infty \frac{\cos(nx)}{x^2+1} \log^m x \, \mathrm{d}x, \quad J_m := \int_0^\infty \frac{\sin(nx)}{x^2+1} \log^m x \, \mathrm{d}x.$$

Corollary 3.2. *For $m \geq 1$,*

$$0 = \sum_{k=0}^{m} \binom{m}{k} \left(\frac{\pi}{2}\right)^k (-1)^{[k/2]} \left\{ \begin{array}{ll} I_{m-k}, & \text{if } k \text{ is even,} \\ J_{m-k}, & \text{if } k \text{ is odd} \end{array} \right\}.$$

We now provide a proof of Theorem 3.1.

Proof. In analogy with our previous proof, we integrate

$$f_s(z) := \frac{e^{inz} z^s}{z^2+1}$$

over the contour $C_{R,\varepsilon}$ and let $\varepsilon \to 0$ and $R \to \infty$. Here, $z^s = e^{s \log z}$ with $-\frac{1}{2}\pi < \arg z \leq \frac{3}{2}\pi$, as above. By the residue theorem,

$$\int_{C_{R,\varepsilon}} f_s(z) \mathrm{d}z = 2\pi i \frac{e^{-n} e^{\pi i s/2}}{2i} = \pi e^{-n} e^{\pi i s/2}. \tag{3.4}$$

Letting $\varepsilon \to 0$ and $R \to \infty$, and using bounds for the integrand on the semi-circles as we did above, we deduce that

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_{C_{R,\varepsilon}} f_s(z) \mathrm{d}z = \int_{-\infty}^\infty \frac{e^{inx} x^s}{x^2+1} \mathrm{d}x = \int_0^\infty \frac{e^{-inx} x^s e^{\pi i s}}{x^2+1} \mathrm{d}x + \int_0^\infty \frac{e^{inx} x^s}{x^2+1} \mathrm{d}x. \tag{3.5}$$

Combining (3.4) and (3.5), we find that

$$\pi e^{-n} e^{\pi i s/2} = \int_0^\infty \{e^{inx} + e^{-inx} e^{\pi i s}\} \frac{x^s}{x^2+1} \mathrm{d}x. \tag{3.6}$$

We then divide both sides of (3.6) by $2e^{\pi i s/2}$ to obtain (3.1). Note that the integrals are absolutely convergent for $s \in (-1, 1)$. By Dirichlet's test, (3.6) holds for $s \in (-1, 2)$. □

Replacing s with $s + 1$ in (3.1), we obtain the following companion integral.

Corollary 3.3. *For $s \in (-2, 1)$ and $n \geq 0$,*

$$\frac{\pi}{2}e^{-n} = \int_0^\infty \frac{x\sin(nx - \pi s/2)}{x^2+1}x^s\mathrm{d}x. \tag{3.7}$$

Example 3.4. Setting $s = 0$ in (3.7), we find that

$$\frac{\pi}{2}e^{-n} = \int_0^\infty \frac{x\sin(nx)}{x^2+1}\mathrm{d}x, \tag{3.8}$$

which is well-known. After taking one derivative with respect to s in (3.7) and setting $s = 0$, we similarly find that

$$0 = \int_0^\infty \frac{x\sin(nx)}{x^2+1}\log x\,\mathrm{d}x - \frac{\pi}{2}\int_0^\infty \frac{x\cos(nx)}{x^2+1}\mathrm{d}x, \tag{3.9}$$

which may be compared with Ramanujan's formula (2.1). As a second example, after taking two derivatives of (3.7) with respect to s, setting $s = 0$, and using (3.8), we arrive at the identity

$$\frac{\pi^3}{8}e^{-n} = \int_0^\infty \frac{x\sin(nx)}{x^2+1}\log^2 x\,\mathrm{d}x - \pi\int_0^\infty \frac{x\cos(nx)}{x^2+1}\log x\,\mathrm{d}x. \tag{3.10}$$

◇

We offer a few additional remarks before generalizing our ideas in the next section. Equating real parts in the identity (3.6) from the proof of Theorem 3.1, we find that

$$\pi e^{-n}\cos(\pi s/2) = \int_0^\infty \{\cos(nx)(1+\cos(\pi s)) + \sin(nx)\sin(\pi s)\}\frac{x^s}{x^2+1}\mathrm{d}x. \tag{3.11}$$

Setting $s = 0$ in (3.11), we again obtain (2.8). On the other hand, note that

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}\{\cos(nx)(1+\cos(\pi s)) + \sin(nx)\sin(\pi s)\}\right]_{s=0} = \pi\sin(nx).$$

Hence, taking a derivative of (3.11) with respect to s, and then setting $s = 0$, we find that

$$0 = \pi\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x + 2\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x,$$

which is the formula (2.1) that Ramanujan recorded on page 391. Similarly, taking two derivatives of (3.11) and letting $s = 0$, we deduce that

$$-\frac{\pi^3}{4}e^{-n} = -\pi^2\int_0^\infty \frac{\cos(nx)}{x^2+1}\mathrm{d}x + 2\pi\int_0^\infty \frac{\sin(nx)}{x^2+1}\log x\,\mathrm{d}x + 2\int_0^\infty \frac{\cos(nx)}{x^2+1}\log^2 x\,\mathrm{d}x,$$

which, using (2.8), simplifies to

$$\frac{\pi^3}{8}e^{-n} = \pi\int_0^\infty \frac{\sin(nx)}{x^2+1}\log x\,\mathrm{d}x + \int_0^\infty \frac{\cos(nx)}{x^2+1}\log^2 x\,\mathrm{d}x$$

which is the formula (2.2) arising from Ramanujan's unintelligible remark in the initial edition of [5].

The integral (3.11) has the companion

$$\pi e^{-n}\sin(\pi s/2) = \int_0^\infty \{\cos(nx)\sin(\pi s) + \sin(nx)(1-\cos(\pi s))\}\frac{x^s}{x^2+1}\mathrm{d}x, \tag{3.12}$$

which is obtained by equating imaginary parts in (3.6). However, taking derivatives of (3.12) with respect to s, and then setting $s = 0$, does not generate new identities. Instead, we recover precisely the previous results. For instance, taking a derivative of (3.12) with respect to s, and then setting $s = 0$, we again deduce (2.8). Taking two derivatives of (3.12) with respect to s, and then setting $s = 0$, we obtain

$$0 = \pi^2\int_0^\infty \frac{\sin(nx)}{x^2+1}\mathrm{d}x + 2\pi\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,\mathrm{d}x,$$

which is again Ramanujan's formula (2.1).

4. General Theorems

The phenomenon observed by Ramanujan in (1.2) can be generalized by replacing the rational function $1/(z^2+1)$ by a general rational function $f(z)$ in which the denominator has degree at least one greater than the degree of the numerator. We shall also assume that $f(z)$ does not have any poles on the real axis. We could prove a theorem allowing for poles on the real axis, but in such instances we would need to consider the principal values of the resulting integrals on the real axis. In our arguments above, we used the fact that $1/(z^2+1)$ is an even function. For our general theorem, we require that $f(z)$ be either even or odd. For brevity, we let $\mathrm{Res}(F(z); z_0)$ denote the residue of a function $F(z)$ at a pole z_0. As above, we define a branch of $\log z$ by $-\frac{1}{2}\pi < \theta = \arg z \leq \frac{3}{2}\pi$.

For a rational function $f(z)$ as prescribed above and each nonnegative integer m, define

$$I_m := \int_0^\infty f(x)\cos x\log^m x\,\mathrm{d}x \quad \text{and} \quad J_m := \int_0^\infty f(x)\sin x\log^m x\,\mathrm{d}x. \tag{4.1}$$

Theorem 4.1. *Let $f(z)$ denote a rational function in z, as described above, and let I_m and J_m be defined by (4.1). Let*

$$S := 2\pi i\sum_U \mathrm{Res}(e^{iz}f(z)\log^m z, z_j), \tag{4.2}$$

where the sum is over all poles z_j of $e^{iz}f(z)\log^m z$ lying in the upper half-plane U. Suppose that $f(z)$ is even. Then

$$S = \sum_{k=0}^m \binom{m}{k}(i\pi)^{m-k}(I_k - iJ_k) + (I_m + iJ_m). \tag{4.3}$$

Suppose that $f(z)$ is odd. Then

$$S = -\sum_{k=0}^m \binom{m}{k}(i\pi)^{m-k}(I_k - iJ_k) + (I_m + iJ_m). \tag{4.4}$$

Observe that (4.3) and (4.4) are recurrence relations that enable us to successively calculate I_m and J_m. With each succeeding value of m, we see that two previously non-appearing integrals arise. If $f(z)$ is even, then these integrals are I_m and J_{m-1}, while if $f(z)$ is odd, these integrals are J_m and I_{m-1}. The previously non-appearing integrals appear in either the real part or the imaginary part of the right-hand sides of (4.3) and (4.4), but not both real and imaginary parts. This fact therefore does not enable us to explicitly determine either of the two integrals. We must be satisfied with obtaining recurrence relations with increasingly more terms.

Proof. We commence as in the proof of Theorem 2.1. Let $C_{R,\varepsilon}$ denote the positively oriented contour consisting of the semi-circle C_R given by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, $[-R, -\epsilon]$, the semi-circle C_ε given by $z = \varepsilon e^{i\theta}$, $\pi \geq \theta \geq 0$, and $[\varepsilon, R]$, where $0 < \varepsilon < d$, where d is the smallest modulus of the poles of $f(z)$ in U. We also choose R larger than the moduli of all the poles of $f(z)$ in U. By the residue theorem,

$$\int_{C_{R,\varepsilon}} e^{iz} f(z) \log^m z \, \mathrm{d}z = S, \tag{4.5}$$

where S is defined in (4.2).

We next directly evaluate the integral on the left-hand side of (4.5). As in the proof of Theorem 2.1, we can easily show that

$$\int_{C_\varepsilon} e^{iz} f(z) \log^m z \, \mathrm{d}z = o(1), \tag{4.6}$$

as ε tends to 0. Secondly, we estimate the integral over C_R. By hypothesis, there exist a positive constant A and a positive number R_0, such that for $R \geq R_0$, $|f(Re^{i\theta})| \leq A/R$. Hence, for $R \geq R_0$,

$$\begin{aligned}\left|\int_{C_R} e^{iz} f(z) \log^m z \, \mathrm{d}z\right| &= \left|\int_0^\pi e^{iRe^{i\theta}} f(Re^{i\theta}) \log^m(Re^{i\theta}) iRe^{i\theta} \mathrm{d}\theta\right| \\ &\leq \int_0^\pi e^{-R\sin\theta} |f(Re^{i\theta})| (\log R + \pi)^m R \, \mathrm{d}\theta \\ &\leq A(\log R + \pi)^m \left(\int_0^{\pi/2} + \int_{\pi/2}^\pi\right) e^{-R\sin\theta} \mathrm{d}\theta. \end{aligned} \tag{4.7}$$

Since $\sin\theta \geq 2\theta/\pi$, $0 \leq \theta \leq \pi/2$, upon replacing θ by $\pi - \theta$, we find that

$$\int_{\pi/2}^\pi e^{-R\sin\theta} \mathrm{d}\theta = \int_0^{\pi/2} e^{-R\sin\theta} \mathrm{d}\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} \mathrm{d}\theta = \frac{\pi}{2R}\left(1 - e^{-R}\right). \tag{4.8}$$

The bound (4.8) also holds for the first integral on the far right-hand side of (4.7). Hence, from (4.7),

$$\left|\int_{C_R} e^{iz} f(z) \log^m z \, \mathrm{d}z\right| \leq A(\log R + \pi)^m \frac{\pi}{R}\left(1 - e^{-R}\right) = o(1), \tag{4.9}$$

as R tends to infinity.

Hence, so far, by (4.5), (4.6), and (4.9), we have shown that

$$\begin{aligned} S &= \int_{-\infty}^{0} e^{ix} f(x)(\log|x| + i\pi)^m \mathrm{d}x + \int_0^{\infty} e^{ix} f(x) \log^m x \,\mathrm{d}x \\ &= \int_0^{\infty} \left\{ e^{-ix} f(-x)(\log x + i\pi)^m + e^{ix} f(x) \log^m x \right\} \mathrm{d}x. \end{aligned} \tag{4.10}$$

Suppose first that $f(x)$ is even. Then (4.10) takes the shape

$$\begin{aligned} S &= \int_0^{\infty} f(x) \left\{ e^{-ix}(\log x + i\pi)^m + e^{ix} \log^m x \right\} \mathrm{d}x \\ &= \int_0^{\infty} f(x) \left\{ e^{-ix} \sum_{k=0}^{m} \binom{m}{k} \log^k x (i\pi)^{m-k} + e^{ix} \log^m x \right\} \mathrm{d}x \\ &= \sum_{k=0}^{m} \binom{m}{k} (i\pi)^{m-k} (I_k - iJ_k) + (I_m + iJ_m), \end{aligned} \tag{4.11}$$

which establishes (4.3). Secondly, suppose that $f(z)$ is odd. Then, (4.10) takes the form

$$\begin{aligned} S &= \int_0^{\infty} f(x) \left\{ -e^{-ix}(\log x + i\pi)^m + e^{ix} \log^m x \right\} \mathrm{d}x \\ &= \int_0^{\infty} f(x) \left\{ -e^{-ix} \sum_{k=0}^{m} \binom{m}{k} \log^k x (i\pi)^{m-k} + e^{ix} \log^m x \right\} \mathrm{d}x \\ &= -\sum_{k=0}^{m} \binom{m}{k} (i\pi)^{m-k} (I_k - iJ_k) + (I_m + iJ_m), \end{aligned} \tag{4.12}$$

from which (4.4) follows. □

Example 4.2. Let $f(z) = z/(z^2+1)$. Then

$$2\pi i \operatorname{Res}\left(\frac{e^{iz} z \log^m z}{z^2+1}, i \right) = \frac{\pi i}{e} \left(\frac{i\pi}{2} \right)^m, \tag{4.13}$$

and so we are led by (4.4) to the recurrence relation

$$\frac{\pi i}{e} \left(\frac{i\pi}{2} \right)^m = -\sum_{k=0}^{m} \binom{m}{k} (i\pi)^{m-k} (I_k - iJ_k) + (I_m + iJ_m), \tag{4.14}$$

where

$$I_m := \int_0^{\infty} \frac{x \cos x \log^m x}{x^2+1} \mathrm{d}x \quad \text{and} \quad J_m := \int_0^{\infty} \frac{x \sin x \log^m x}{x^2+1} \mathrm{d}x.$$

(In the sequel, it is understood that we are assuming that $n = 1$ in Theorem 2.1 and in all our deliberations of the two preceding sections.) If $m = 0$, (4.14) reduces to

$$J_0 = \frac{\pi}{2e}, \tag{4.15}$$

which is (2.8). After simplification, if $m=1$, (4.14) yields

$$-\frac{\pi^2}{2e} = -i\pi I_0 - \pi J_0 + 2iJ_1. \tag{4.16}$$

If we equate real parts in (4.16), we once again deduce (4.15). If we equate imaginary parts in (4.16), we find that

$$J_1 - \frac{\pi}{2} I_0 = 0, \tag{4.17}$$

which is identical with (3.9). Setting $m=2$ in (4.14), we find that

$$-\frac{i\pi^3}{4e} = \pi^2(I_0 - iJ_0) - 2i\pi(I_1 - iJ_1) + 2iJ_2. \tag{4.18}$$

Equating real parts on both sides of (4.18), we once again deduce (4.17). If we equate imaginary parts in (4.18) and employ (4.15), we arrive at

$$J_2 - \pi I_1 = \frac{\pi^3}{8e}, \tag{4.19}$$

which is the same as (3.10). Lastly, we set $m=3$ in (4.14) to find that

$$\frac{\pi^4}{8e} = i\pi^3(I_0 - iJ_0) + 3\pi^2(I_1 - iJ_1) - 3i\pi(I_2 - iJ_2) + 2iJ_3. \tag{4.20}$$

If we equate real parts on both sides of (4.20) and simplify, we deduce (4.19) once again. On the other hand, when we equate imaginary parts on both sides of (4.20), we deduce that

$$2J_3 - 3\pi I_2 - 3\pi^2 J_1 + \pi^3 I_0 = 0. \tag{4.21}$$

A slight simplification of (4.21) can be rendered with the use of (4.17). ◇

We can replace the rational function $1/(x^2+1)$ in Theorem 3.1 by other even rational functions $f(x)$ to obtain the following generalization of Theorem 3.1. Its proof is in the same spirit as that of Theorem 4.1.

Theorem 4.3. *Suppose that $f(z)$ is an even rational function with no real poles and with the degree of the denominator exceeding the degree of the numerator by at least 2. Then,*

$$\frac{\pi i}{e^{\pi i s/2}} \sum_U \operatorname{Res}(e^{inz} f(z) z^s, z_j) = \int_0^\infty \cos(nx - \pi s/2) f(x) x^s \mathrm{d}x,$$

where the sum is over all poles z_j of $f(z)$ lying in the upper half-plane U.

Note that, as we did for (3.7), we can replace s with $s+1$ in Theorem 4.3 to obtain a corresponding result for odd rational functions $xf(x)$. This is illustrated in Example 4.7 below.

As an application, we derive from Theorem 4.3 the following explicit integral evaluation, which reduces to Theorem 3.1 when $r=0$.

Theorem 4.4. *Let $r \geq 0$ be an integer. For $s \in (-1, 2(r+1))$ and $n \geq 0$,*

$$\int_0^\infty \frac{\cos(nx - \pi s/2)}{(x^2+1)^{r+1}} x^s \mathrm{d}x = \frac{\pi}{2} e^{-n} \sum_{k=0}^{r} \frac{1}{2^{r+k}} \binom{r+k}{k} \sum_{j=0}^{r-k} (-1)^j \binom{s}{j} \frac{n^{r-k-j}}{(r-k-j)!}.$$

Proof. Setting $f(z) = 1/(z^2+1)^r$ in Theorem 4.3, we see that we need to calculate the residue

$$\operatorname{Res}\left(\frac{e^{inz}z^s}{(z^2+1)^{r+1}}, i\right) = \operatorname{Res}\left(\frac{\alpha(z)}{(z-i)^{r+1}}, i\right),$$

where

$$\alpha(z) = \frac{e^{inz}z^s}{(z+i)^{r+1}}$$

is analytic in a neighborhood of $z = i$. Equivalently, we calculate the coefficient of x^r in the Taylor expansion of $\alpha(x+i)$ around $x = 0$. Using the binomial series

$$\frac{1}{(x+a)^{r+1}} = \sum_{k\geq 0}(-1)^k\binom{r+k}{k}x^k a^{-r-k-1}$$

with $a = 2i$, we find that

$$\begin{aligned}\alpha(x+i) &= e^{-n}\frac{e^{inx}(x+i)^s}{(x+2i)^{r+1}}\\ &= e^{-n}\sum_{k\geq 0}(-1)^k\binom{r+k}{k}x^k(2i)^{-r-k-1}\sum_{j\geq 0}\binom{s}{j}x^j i^{s-j}\sum_{l\geq 0}\frac{(inx)^l}{l!}.\end{aligned}$$

Extracting the coefficient of x^r, we conclude that

$$\begin{aligned}\operatorname{Res}\left(\frac{e^{inz}z^s}{(z^2+1)^{r+1}}, i\right) &= \frac{e^{-n}}{(2i)^{r+1}}\sum_{k=0}^{r}\frac{(-1)^k}{(2i)^k}\binom{r+k}{k}\sum_{j=0}^{r-k}\binom{s}{j}i^{s-j}\frac{(in)^{r-k-j}}{(r-k-j)!}\\ &= \frac{e^{-n}e^{\pi is/2}}{2i}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\sum_{j=0}^{r-k}(-1)^j\binom{s}{j}\frac{n^{r-k-j}}{(r-k-j)!}.\end{aligned}$$

Theorem 4.4 now follows from Theorem 4.3. □

Example 4.5. In particular, in the case $s = 0$,

$$\int_0^\infty \frac{\cos(nx)}{(x^2+1)^{r+1}}\mathrm{d}x = \frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\frac{n^{r-k}}{(r-k)!}. \tag{4.22}$$

We note that, more generally, this integral can be expressed in terms of the modified Bessel function $K_{r+1/2}(z)$ of order $r + 1/2$. Namely, we have [3, formula 3.773, no. 6]

$$\int_0^\infty \frac{\cos(nx)}{(x^2+1)^{r+1}}\mathrm{d}x = \left(\frac{n}{2}\right)^{r+1/2}\frac{\sqrt{\pi}}{\Gamma(r+1)}K_{r+1/2}(n). \tag{4.23}$$

When $r \geq 0$ is an integer, the Bessel function $K_{r+1/2}(z)$ is elementary and the right-hand side of (4.23) evaluates to the right-hand side of (4.22). ◇

On the other hand, taking a derivative with respect to s before setting $s = 0$, and observing that, for $j \geq 1$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}\binom{s}{j}\right]_{s=0} = \frac{(-1)^{j-1}}{j},$$

we arrive at the following generalization of Ramanujan's formula (3.2).

Corollary 4.6. *We have*

$$\begin{aligned}&\int_0^\infty \frac{\cos(nx)}{(x^2+1)^{r+1}}\log x\,\mathrm{d}x+\frac{\pi}{2}\int_0^\infty \frac{\sin(nx)}{(x^2+1)^{r+1}}\mathrm{d}x\\ = \quad & -\frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\sum_{j=1}^{r-k}\frac{1}{j}\frac{n^{r-k-j}}{(r-k-j)!}.\end{aligned}$$

We leave it to the interested reader to make explicit the corresponding generalization of (3.3).

Example 4.7. As a direct extension of (3.7), replacing s with $s+1$ in Theorem 4.4, we obtain the following companion integral. For integers $r \geq 0$, and any $s \in (-2, 2r+1)$ and $n \geq 0$,

$$\int_0^\infty \frac{x\sin(nx-\pi s/2)}{(x^2+1)^{r+1}}x^s\mathrm{d}x=\frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\sum_{j=0}^{r-k}(-1)^j\binom{s+1}{j}\frac{n^{r-k-j}}{(r-k-j)!}.$$

In particular, setting $s=0$, we find that

$$\int_0^\infty \frac{x\sin(nx)}{(x^2+1)^{r+1}}\mathrm{d}x=\frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\left\{\frac{n^{r-k}}{(r-k)!}-\frac{n^{r-k-1}}{(r-k-1)!}\right\}, \tag{4.24}$$

while taking a derivative with respect to s before setting $s=0$ and observing that, for $j \geq 2$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}s}\binom{s+1}{j}\right]_{s=0}=\frac{(-1)^j}{j(j-1)},$$

we find that

$$\begin{aligned}&\int_0^\infty \frac{x\sin(nx)}{(x^2+1)^{r+1}}\log x\,\mathrm{d}x-\frac{\pi}{2}\int_0^\infty \frac{x\cos(nx)}{(x^2+1)^{r+1}}\mathrm{d}x\\ = \quad & \frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\left[-\frac{n^{r-k-1}}{(r-k-1)!}+\sum_{j=2}^{r-k}\frac{1}{j(j-1)}\frac{n^{r-k-j}}{(r-k-j)!}\right]\\ = \quad & \frac{\pi}{2}e^{-n}\sum_{k=0}^{r}\frac{1}{2^{r+k}}\binom{r+k}{k}\sum_{j=2}^{r-k}\frac{1}{j(j-1)}\frac{n^{r-k-j}}{(r-k-j)!}\\ & +\int_0^\infty \frac{x\sin(nx)}{(x^2+1)^{r+1}}\mathrm{d}x-\int_0^\infty \frac{\cos(nx)}{(x^2+1)^{r+1}}\mathrm{d}x,\end{aligned}$$

upon the employment of (4.22) and (4.24). ◇

References

[1] B. C. Berndt, *Ramanujan's Notebooks*, Part I, Springer-Verlag, New York, 1985.
[2] B. C. Berndt, *Ramanujan's Notebooks*, Part IV, Springer-Verlag, New York, 1994.
[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, edited by D. Zwillinger and V. H. Moll, Academic Press, San Diego, 8th ed., 2014.
[4] S. Ramanujan, *Collected Papers*, Cambridge University Press, Cambridge, 1927; reprinted by Chelsea, New York, 1962; reprinted by the American Mathematical Society, Providence, RI, 2000.

[5] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957; second ed., 2012.

Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 W. Green St., Urbana, IL 61801, USA

E-mail address: berndt@illinois.edu

Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 W. Green St., Urbana, IL 61801, USA

Current address: Department of Mathematics and Statistics, University of South Alabama, 411 University Blvd N, Mobile, AL 36688, USA

E-mail address: straub@southalabama.edu