A Sinc that Sank

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Abstract

We resolve and further study a sinc integral evaluation, first posed in this MONTHLY in [1967, p. 1015], which was solved in [1968, p. 914] and withdrawn in [1970, p. 657]. After a short introduction to the problem and its history, we give a general evaluation which we make entirely explicit in the case of the product of three sinc functions. Finally, we exhibit some more general structure of the integrals in question.

Key words: sinc function, conditionally convergent integral, numeric-symbolic computation.

1 Introduction and background

In [1967, #5529, p. 1015] D. Mitrinović asked in this MONTHLY for an evaluation of

$$I_n := \int_{-\infty}^{\infty} \prod_{j=1}^{n} \frac{\sin\left(k_j(x-a_j)\right)}{x-a_j}\,\mathrm{d}x \tag{1}$$

for real numbers a_j, k_j with $1 \leqslant j \leqslant n$. We shall write $I_n\binom{a_1,\ldots,a_n}{k_1,\ldots,k_n}$ when we wish to emphasize the dependence on the parameters. Up to a constant factor, (1) is an integral over a product of sinc functions: $\operatorname{sinc} x := \frac{\sin x}{x}$.

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The next year a solution [1968, #5529, p. 914] was published in the form of

$$I_n = \pi \prod_{j=2}^{n} \frac{\sin\left(k_j(a_1 - a_j)\right)}{a_1 - a_j} \tag{2}$$

under the assumption that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n \geqslant 0$. This solution as M. Klamkin pointed out in [1970, p. 657] can not be correct, since it is not symmetric in the parameters while I_n is. Indeed in the case $k_1 = k_2 = k_3 = 1$ the evaluation (2) is not symmetric in the variables a_j and gives differing answers on permuting the a's. The proof given relies on formally correct Fourier analysis; but there are missing constraints on the k_j variables which have the effect that it is seldom right for more than two variables. Indeed, as shown then by D. Djoković and L. Glasser [10] — who were both working in Waterloo at the time — the evaluation (2) holds true under the restriction $k_1 \geqslant k_2 + k_3 + \ldots + k_n$ when all of the k_j are positive. However, no simple general fix appeared possible — and indeed for $n > 2$ the issue is somewhat complex. The problem while recorded several times in later MONTHLY lists of unsolved problems appears (from a JSTOR hunt[1]) to have disappeared without trace in the later 1980's.

The precise issues regarding evaluation of sinc integrals are described in detail in [4] or [5, Chapter 2] along with some remarkable consequences [2, 4, 5]. In the two-variable case the 1968 solution is essentially correct: we do obtain

$$I_2 = \pi \frac{\sin\left((k_1 \wedge k_2)(a_1 - a_2)\right)}{a_1 - a_2} \tag{3}$$

for $a_1 \neq a_2$ as will be made explicit below. Here $a \wedge b := \min\{a, b\}$. Some of the delicacy is a consequence of the fact that the classical sinc evaluation given next is only conditionally true [4]. We have

$$\int_{-\infty}^{\infty} \frac{\sin(kx)}{x} \, \mathrm{d}x = \pi \operatorname{sgn}(k), \tag{4}$$

where $\operatorname{sgn}(0) = 0, \operatorname{sgn}(k) = 1$ for $k > 0$ and $\operatorname{sgn}(k) = -1$ for $k < 0$.

In (4) the integral is absolutely divergent and is best interpreted as a Cauchy-Riemann limit. Thus the evaluation of (1) yields $I_1 = \pi \operatorname{sgn}(k_1)$ which has a discontinuity at $k_1 = 0$. For $n \geqslant 2$, however, I_n is an absolutely convergent integral which

[1] A search on JSTOR through all MONTHLY volumes, suggests that the solutions were never published and indeed for some years the original problem reappeared on lists of unsolved MONTHLY problems before apparently disappearing from view. Such a JSTOR search is not totally convincing since there is no complete indexing of problems and their status.

is (jointly) continuous as a function of all k_j and all a_j. This follows from Lebesgue's dominated convergence theorem since the absolute value of the integrand is less than $\prod_{j=1}^n |k_j|$ for all x and less than $2/x^2$ for all sufficiently large $|x|$.

It is worth observing that the oscillatory structure of the integrals, see Figure 1, means that their evaluation both numerically and symbolically calls for a significant amount of care.

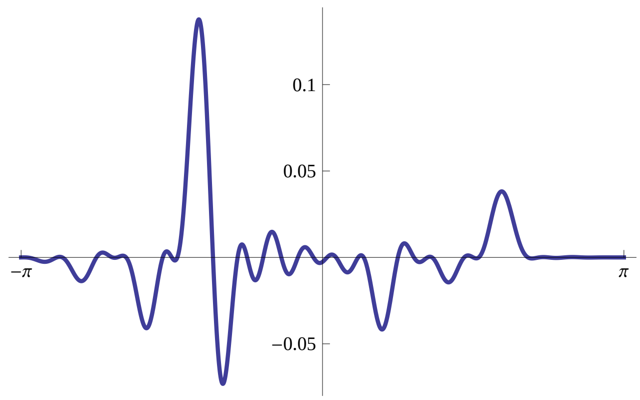

Figure 1: Integrand in (1) with $\mathbf{a} = (-3, -2, -1, 0, 1, 2)$, $\mathbf{k} = (1, 2, 3, 4, 5, 6)$

We wish to emphasize the continuing fascination with similar sinc integrals [11]. Indeed, the work in [4] was triggered by the exact problem described by K. Morrison in this MONTHLY [14]. This led also to a lovely MONTHLY article on random series [15], and there is further related work in [6].

A most satisfactory introduction to the many applications and properties of the sinc function is given in [9]. Additionally, T. Feeman's recent book on medical imaging [7] chose to begin with the example given at the beginning of Section 4. The paper [12] makes a careful and historically informed study of the geometric results implicit in the study of related multiple sinc integrals. This includes recording that G. Pólya in his 1912 doctoral thesis showed that if $k = (k_1, \ldots, k_n)$ has non-zero coefficients and

$$S_k(\theta) := \{x \in \mathbb{R}^n \ : \ |\langle k, x\rangle| \leqslant \theta/2, \, x \in C^n\}$$

denotes the slab inside the hypercube $C^n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ cut off by the hyperplanes $\langle k, x\rangle = \pm\theta/2$, then

$$\operatorname{Vol}_n(S_k(\theta)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\theta x)}{x} \prod_{j=1}^{n} \frac{\sin(k_j x)}{k_j x} \, \mathrm{d}x, \tag{5}$$

a relationship we return to in Section 4. More general polyhedra volumes determined by multidimensional sinc integrals are examined in [3]. As a consequence of (5) and described for instance in [4], the integral (5) may also be interpreted as the probability that

$$\left|\sum_{j=1}^{n} k_j X_j\right| \leqslant \theta \tag{6}$$

where X_j are independent random variables uniformly distributed on $[-1, 1]$.

2 Evaluation of I_n

Without loss of generality we assume that all k_j are strictly positive. In this section we shall only consider the case when all the a_j are distinct. As illustrated in Sections 3.2 and 3.3 the special cases can be treated by taking limits. We begin with the classical and simple partial fraction expression

$$\prod_{j=1}^{n} \frac{1}{x-a_j} = \sum_{j=1}^{n} \frac{1}{x-a_j} \prod_{i \neq j} \frac{1}{a_j - a_i} \tag{7}$$

valid when the a_j are distinct. Applying (7) to the integral I_n we then have

$$\begin{aligned} I_n &= \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{\sin(k_j(x-a_j))}{x-a_j} \prod_{i \neq j} \frac{\sin(k_i(x-a_i))}{a_j - a_i} \,\mathrm{d}x \\ &= \sum_{j=1}^{n} \int_{-\infty}^{\infty} \frac{\sin(k_j x)}{x} \prod_{i \neq j} \frac{\sin(k_i(x+(a_j-a_i)))}{a_j - a_i} \,\mathrm{d}x. \end{aligned} \tag{8}$$

We pause and illustrate the general approach in the case of $n = 2$ variables.

Example 2.1 (Two variables)**.** We apply (8) to write

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} \frac{\sin(k_1 x)}{x} \frac{\sin\left(k_2(x+a_1-a_2)\right)}{a_2-a_1} \,\mathrm{d}x + \int_{-\infty}^{\infty} \frac{\sin(k_2 x)}{x} \frac{\sin\left(k_1(x+a_2-a_1)\right)}{a_1-a_2} \,\mathrm{d}x \\ &= \frac{\sin\left(k_2(a_1-a_2)\right)}{a_1-a_2} \int_{-\infty}^{\infty} \frac{\sin(k_1 x)}{x} \cos(k_2 x) \,\mathrm{d}x + \frac{\sin\left(k_1(a_2-a_1)\right)}{a_2-a_1} \int_{-\infty}^{\infty} \frac{\sin(k_2 x)}{x} \cos(k_1 x) \,\mathrm{d}x \end{aligned}$$

where for the second equation we have used the addition formula for the sine and noticed that the sine-only integrands (being odd) integrate to zero. Finally, we either appeal to [4, Theorem 3] or express

$$\int_{-\infty}^{\infty} \frac{\sin(k_1 x)}{x} \cos(k_2 x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\left((k_1+k_2)x\right)}{x}\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\left((k_1-k_2)x\right)}{x}\,\mathrm{d}x,$$
$$\int_{-\infty}^{\infty} \frac{\sin(k_2 x)}{x} \cos(k_1 x)\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\left((k_1+k_2)x\right)}{x}\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\left((k_1-k_2)x\right)}{x}\,\mathrm{d}x,$$

and appeal twice to (4) to obtain the final elegant cancellation

$$I_2 = \pi\,\frac{\sin\left((k_1 \wedge k_2)(a_1-a_2)\right)}{a_1-a_2} \tag{9}$$

valid for $a_1 \neq a_2$. We observe that the result remains true for $a_1 = a_2$, in which case the right-hand side of (9) attains the limiting value $\pi(k_1 \wedge k_2)$. $\Diamond$

Let us observe that after the first step in Example 2.1 — independent of the exact final formula — the integrals to be obtained have lost their dependence on the a_j. This is what we exploit more generally. Proceeding as in Example 2.1 and applying the addition formula to (8) we write:

$$I_n = \sum_{j=1}^{n} \sum_{A,B} C_{j,A,B}\; \alpha_{j,A,B} \tag{10}$$

where the sum is over all sets A and B partitioning $\{1, 2\ldots, j-1, j+1, \ldots n\}$, and

$$C_{j,A,B} := \prod_{i \in A} \frac{\cos(k_i(a_j - a_i))}{a_j - a_i} \prod_{i \in B} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i} \tag{11}$$

while

$$\alpha_{j,A,B} := \int_{-\infty}^{\infty} \prod_{i \in A \cup \{j\}} \sin(k_i x) \prod_{i \in B} \cos(k_i x)\,\frac{\mathrm{d}x}{x}. \tag{12}$$

Notice that we may assume the cardinality $|A|$ of A to be even since the integral in (12) vanishes if $|A|$ is odd.

To further treat (10) we write the products of sines and cosines in terms of sums of single trigonometric functions. The general formulae are made explicit next.

Proposition 2.2 (Cosine Product)**.**

$$\prod_{j=1}^{r} \cos(x_j) = 2^{-r} \sum_{\varepsilon \in \{-1,1\}^r} \cos\left(\sum_{j=1}^{r} \varepsilon_j x_j\right). \tag{13}$$

Proof. The formula follows inductively from the trigonometric identity $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$. □

Observe that by taking derivatives with respect to some of the x_j in (13) we obtain similar formulae for general products of sines and cosines.

Corollary 2.3 (Sine and Cosine Product)**.**

$$\prod_{j=1}^{s} \sin(x_j) \prod_{j=s+1}^{r} \cos(x_j) = 2^{-r} \sum_{\varepsilon\in\{-1,1\}^r} \left(\prod_{j=1}^{s} \varepsilon_i\right) \cos\left(\sum_{j=1}^{r} \varepsilon_j x_j - \frac{s\pi}{2}\right). \tag{14}$$

Here, we used $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = \cos(x-\pi/2)$ to write the evaluation rather compactly. Note that when $s = 2k+1$ is odd then $\cos(x - s\pi/2) = (-1)^k \sin(x)$. Applying (14) to the definition (12), it thus follows that, for even $|A|$,

$$\begin{aligned} \alpha_{j,A,B} &= (-1)^{|A|/2} \int_{-\infty}^{\infty} \frac{1}{2^n} \sum_{\varepsilon\in\{-1,1\}^n} \left(\prod_{i\in A\cup\{j\}} \varepsilon_i\right) \sin\left(\sum_{i=1}^{n} \varepsilon_i k_i x\right) \frac{\mathrm{d}x}{x} \\ &= \pi\,(-1)^{|A|/2} \frac{1}{2^n} \sum_{\varepsilon\in\{-1,1\}^n} \left(\prod_{i\in A\cup\{j\}} \varepsilon_i\right) \operatorname{sgn}\left(\sum_{i=1}^{n} \varepsilon_i k_i\right). \end{aligned} \tag{15}$$

Then on combining (15) with (10) we obtain the following general evaluation:

Theorem 2.4 (General Evaluation)**.** *We have*

$$I_n = \sum_{j=1}^{n} \sum_{A,B} \alpha_{j,A,B}\, C_{j,A,B} \tag{16}$$

where the inner summation is over disjoint sets A, B such that $|A|$ is even and $A \cup B = \{1, 2, \ldots, j-1, j+1, \ldots, n\}$. The trigonometric products

$$C_{j,A,B} = \prod_{i\in A} \frac{\cos(k_i(a_j - a_i))}{a_j - a_i} \prod_{i\in B} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i}$$

are as in (11) and $\alpha_{j,A,B}$ is given by (15).

Note that in dimension $n \geqslant 2$, there are $n2^{n-2}$ elements $C_{j,A,B}$ which may or may not be distinct.

Remark 2.5. Note that, just like for the defining integral for I_n, it is apparent that the terms $C_{j,A,B}$ and hence the evaluation of I_n given in (16) only depend on the parameters a_j up to a common shift. In particular, setting $b_j = a_j - a_n$ for $j = 1, \ldots, n-1$ the evaluation in (16) can be written as a symmetric function in the $n-1$ variables b_j. $\lozenge$

As an immediate consequence of Theorem 2.4 we have:

Corollary 2.6 (Simplest Case)**.** *Assume, without loss, that $k_1, k_2, \ldots, k_n > 0$. Suppose that there is an index ℓ such that*

$$k_\ell > \frac{1}{2} \sum k_i.$$

In that case, the original solution to the Monthly *problem is valid; that is,*

$$I_n = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \frac{\sin(k_i(x-a_i))}{x-a_i} \mathrm{d}x = \pi \prod_{i \neq \ell} \frac{\sin(k_i(a_\ell - a_i))}{a_\ell - a_i}.$$

This result was independently obtained by Djoković and Glasser [10].

Proof. In this case,

$$\operatorname{sgn}\left(\sum_{i=1}^{n} \varepsilon_i k_i\right) = \varepsilon_\ell$$

for all values of the ε_i. The claim now follows from Theorem 2.4. More precisely, if there is some index $k \neq \ell$ such that $k \in A$ or $j = k$ then $\alpha_{j,A,B} = 0$. This is because the term in (15) contributed by $\varepsilon \in \{-1, 1\}^n$ has opposite sign than the term contributed by ε', where ε' is obtained from ε by flipping the sign of ε_k. It remains to observe that $\alpha_{\ell,\emptyset,B} = \pi$ according to (15). This is the only surviving term. $\square$

2.1 Alternative evaluation of I_n

In 1970 Djoković sent in a solution to the Monthly after the original solution was withdrawn [10]. He used the following identity involving the principal value (PV) of the integral

$$(\mathrm{PV}) \int_{-\infty}^{\infty} \frac{e^{itx}}{x-a_j} \mathrm{d}x = \lim_{\delta \to 0+} \left\{ \int_{-\infty}^{a_j - \delta} + \int_{a_j + \delta}^{\infty} \right\} \frac{e^{itx}}{x-a_j} \mathrm{d}x = \pi i \operatorname{sgn}(t) e^{ita_j} \tag{17}$$

where t is real. Note that setting $a_j = 0$ and taking the imaginary part of (17) gives (4). He then showed, using the same partial fraction expansion as above, that

$$I_n = \frac{\pi i}{(2i)^n} \sum_{j=1}^{n} \left\{ A_j \sum_{\varepsilon \in \{-1,1\}^n} \left(\prod_{r=1}^{n} \varepsilon_r \right) \operatorname{sgn} \left(\sum_{r=1}^{n} \varepsilon_r k_r \right) \exp \left(i \sum_{r=1}^{n} \varepsilon_r k_r (a_j - a_r) \right) \right\} \tag{18}$$

where $a_1, a_2, \ldots, a_n$ are distinct and

$$A_j := \frac{1}{\prod_{r \neq j} (a_j - a_r)}. \tag{19}$$

The formula (18) is quite elegant and also allows one to derive Corollary 2.6, which was independently found by Glasser [10]. For instance, it suffices to appeal to the case $r = s$ of (14). However, as we will demonstrate in the remainder, the evaluation given in Theorem 2.4 has the advantage of making significant additional structure of the integrals I_n more apparent. Before doing so in Section 5 we next consider the case I_3 in detail.

3 The case $n = 3$

We can completely dispose of the three-dimensional integral I_3 by considering the three cases: a_1, a_2, a_3 distinct; a_1 distinct from $a_2 = a_3$; and $a_1 = a_2 = a_3$.

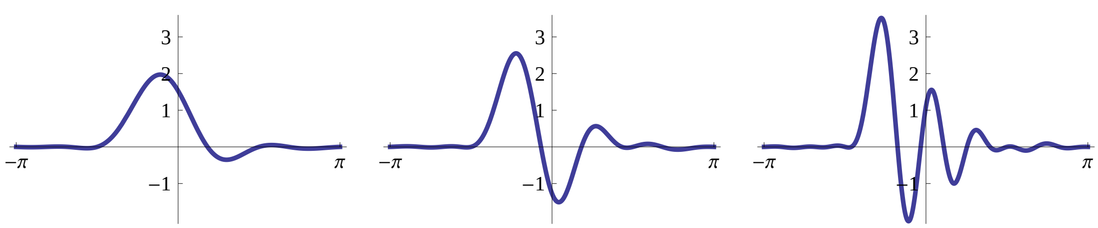

Figure 2: Integrands in (1) with parameters $\mathbf{a} = (-1, 0, 1)$ and $\mathbf{k} = (k_1, 2, 1)$ where $k_1 = 2, 4, 7$

3.1 The case $n = 3$ when a_1, a_2, a_3 are distinct

As demonstrated in this section, the evaluation of I_3 will depend on which inequalities are satisfied by the parameters k_1, k_2, k_3. For $n = 3$, Theorem 2.4 yields:

$$I_3\begin{pmatrix} a_1, a_2, a_3 \\ k_1, k_2, k_3 \end{pmatrix} = \frac{1}{8}\sum_{j=1}^{3}\sum_{\varepsilon\in\{-1,1\}^3}\Bigg[\varepsilon_j \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3)\prod_{i\neq j}\frac{\sin(k_i(a_j - a_i))}{a_j - a_i}$$
$$- \varepsilon_1\varepsilon_2\varepsilon_3 \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3)\prod_{i\neq j}\frac{\cos(k_i(a_j - a_i))}{a_j - a_i}\Bigg]. \quad (20)$$

Remark 3.1 (Recovering Djoković's Evaluation)**.** Upon using the identity $\sin(x)\sin(y) - \cos(x)\cos(y) = -\cos(x+y)$ to combine the two products, the right-hand side of equation (20) can be reexpressed in the symmetric form

$$-\frac{1}{8}\sum_{\varepsilon\in\{-1,1\}^3}\varepsilon_1\varepsilon_2\varepsilon_3 \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3)\sum_{j=1}^{3}\frac{\cos(\sum_{i\neq j}\varepsilon_i k_i(a_j - a_i))}{\prod_{i\neq j}(a_j - a_i)}.$$

This is precisely Djoković's evaluation (18). ◊

In fact, distinguishing between two cases, illustrated in Figure 3, the evaluation (20) of I_3 can be made entirely explicit:

Corollary 3.2 (a_1, a_2, a_3 distinct)**.** *Assume that* $k_1, k_2, k_3 > 0$. *Then*

1. *If* $\frac{1}{2}\sum k_i \leqslant k_\ell$, *as can happen for at most one index* ℓ, *then*

$$I_3\begin{pmatrix} a_1, a_2, a_3 \\ k_1, k_2, k_3 \end{pmatrix} = \pi\prod_{i\neq \ell}\frac{\sin(k_i(a_\ell - a_i))}{a_\ell - a_i}. \quad (21)$$

2. *Otherwise, that is if* $\max k_i < \frac{1}{2}\sum k_i$, *then*

$$I_3\begin{pmatrix} a_1, a_2, a_3 \\ k_1, k_2, k_3 \end{pmatrix} = \frac{\pi}{2}\sum_{j=1}^{3}\left[\prod_{i\neq j}\frac{\sin(k_i(a_j - a_i))}{a_j - a_i} + \prod_{i\neq j}\frac{\cos(k_i(a_j - a_i))}{a_j - a_i}\right]. \quad (22)$$

Proof. The first case is a special case of Corollary 2.6. Alternatively, assuming without loss that the inequality for k_ℓ is strict, it follows directly from (20): because $\operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3) = \varepsilon_\ell$ all but one sum over $\varepsilon \in \{-1, 1\}^3$ cancel to zero.

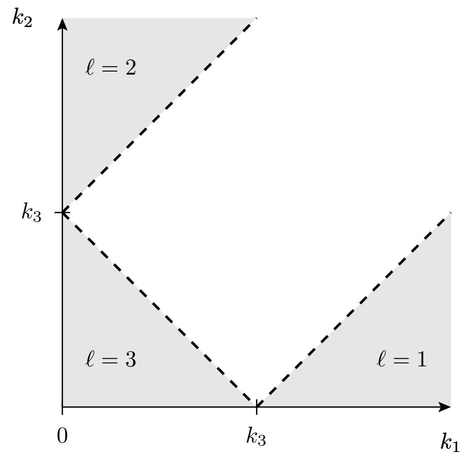

Figure 3: The constraints on the parameters k_j in Corollary 3.2 — if k_1, k_2 take values in the shaded regions then (21) holds with the indicated choice of ℓ

In the second case, $k_1 < k_2 + k_3$, $k_2 < k_3 + k_1$, $k_3 < k_1 + k_2$. Therefore

$$\frac{1}{8} \sum_{\varepsilon \in \{-1,1\}^3} \varepsilon_j \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3) = \frac{1}{2} \quad \text{for all } j,$$

$$-\frac{1}{8} \sum_{\varepsilon \in \{-1,1\}^3} \varepsilon_1 \varepsilon_2 \varepsilon_3 \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3) = \frac{1}{2}.$$

The claim then follows from (20). □

Remark 3.3 (Hidden Trigonometric Identities)**.** Observe that because of the continuity of I_3 as a function of k_1, k_2, and k_3, we must have the non-obvious identity

$$\prod_{i \neq 1} \left[\frac{\sin(k_i(a_1 - a_i))}{a_1 - a_i} \right] = \frac{1}{2} \sum_{j=1}^{3} \left[\prod_{i \neq j} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i} + \prod_{i \neq j} \frac{\cos(k_i(a_j - a_i))}{a_j - a_i} \right] \tag{23}$$

when $k_1 = k_2 + k_3$. We record that *Mathematica 7* is able to verify (23); however, it struggles with the analogous identities arising for $n \geqslant 4$. ♢

3.2 The case $n = 3$ when $a_1 \neq a_2 = a_3$

As a limiting case of Corollary 3.2 we obtain:

Corollary 3.4 ($a_1 \neq a_2 = a_3$)**.** *Assume that $k_1, k_2, k_3 > 0$ and $a_1 \neq a_2$. Set $a := a_2 - a_1$.*

1. *If $k_1 \geqslant \frac{1}{2} \sum k_i$ then*

$$I_3 \binom{a_1, a_2, a_2}{k_1, k_2, k_3} = \pi \frac{\sin(k_2 a)}{a} \frac{\sin(k_3 a)}{a}. \tag{24}$$

2. *If $\max(k_2, k_3) \geqslant \frac{1}{2} \sum k_i$ then*

$$I_3 \binom{a_1, a_2, a_2}{k_1, k_2, k_3} = \pi \min(k_2, k_3) \frac{\sin(k_1 a)}{a}. \tag{25}$$

3. *Otherwise, that is if $\max k_i < \frac{1}{2} \sum k_i$*

$$I_3 \binom{a_1, a_2, a_2}{k_1, k_2, k_3} = \frac{\pi}{2} \frac{\cos((k_2 - k_3)a) - \cos(k_1 a)}{a^2} + \frac{\pi}{2} \frac{(k_2 + k_3 - k_1) \sin(k_1 a)}{a}. \tag{26}$$

Proof. The first two cases are immediate consequences of (21) upon taking the limit $a_3 \to a_2$.

Likewise, the third case follows from (22) with just a little bit of care. The contribution of the sine products from (22) is

$$\frac{\pi}{2} \frac{\sin(k_2 a) \sin(k_3 a)}{a^2} + \frac{\pi}{2} \frac{(k_2 + k_3) \sin(k_1 a)}{a}.$$

On the other hand, writing $a_3 = a_2 + \varepsilon$ with the intent of letting $\varepsilon \to 0$, the cosine products contribute

$$\frac{\pi}{2} \left[\frac{\cos(k_2 a) \cos(k_3 a)}{a^2} - \frac{\cos(k_1 a) \cos(k_3 \varepsilon)}{a \varepsilon} + \frac{\cos(k_1 (a + \varepsilon)) \cos(k_2 \varepsilon)}{(a + \varepsilon)\varepsilon} \right].$$

The claim therefore follows once we show

$$\frac{\cos(k_1 a) \cos(k_3 \varepsilon)}{a \varepsilon} - \frac{\cos(k_1 (a + \varepsilon)) \cos(k_2 \varepsilon)}{(a + \varepsilon)\varepsilon} \to \frac{\cos(k_1 a)}{a^2} + \frac{k_1 \sin(k_1 a)}{a}.$$

This is easily verified by expanding the left-hand side in a Taylor series with respect to ε. In fact, all the steps in this proof can be done automatically using, for instance, *Mathematica 7*. $\square$

Observe that, since I_n is invariant under changing the order of its arguments, Corollary 3.4 covers all cases where exactly two of the parameters a_j agree.

Remark 3.5 (Alternative Approach)**.** We remark that Corollary 3.4 can alternatively be proved in analogy with the proof given for Theorem 2.4 — that is by starting with a partial fraction decomposition and evaluating the occurring basic integrals. Besides integrals covered by equation (15) this includes formulae such as

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\sin(k_2 x)}{x} \frac{\sin(k_3 x)}{x} \cos(k_1 x) \mathrm{d}x \\ &\qquad = \frac{\pi}{8} \sum_{\varepsilon \in \{-1,1\}^3} \varepsilon_2 \varepsilon_3 (\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3) \operatorname{sgn}(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3). \end{aligned} \tag{27}$$

This evaluation follows from [4, Theorem 3(ii)]. In fact, (27) is an immediate consequence of equation (15) with $n = 3$ and $A = \emptyset$ after integrating with respect to one of the parameters k_i where $i \in B$. Clearly, this strategy evaluates a large class of integrals, similar to (27), over the real line with integrands products of sines and cosines as well as powers of the integration variable (see also [4]). ♢

3.3 The case $n = 3$ when $a_1 = a_2 = a_3$

In this case,

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \frac{\sin(k_1(x-a_1))}{x-a_1} \frac{\sin(k_2(x-a_1))}{x-a_1} \frac{\sin(k_3(x-a_1))}{x-a_1} \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\sin(k_1 x)}{x} \frac{\sin(k_2 x)}{x} \frac{\sin(k_3 x)}{x} \mathrm{d}x. \end{aligned} \tag{28}$$

Corollary 3.6 ($a_1 = a_2 = a_3$)**.** *Assume without loss that $k_1 \geqslant k_2 \geqslant k_3 > 0$. Then*

1. *If $k_1 \geqslant k_2 + k_3$ then*
$$I_3 \begin{pmatrix} a_1, a_1, a_1 \\ k_1, k_2, k_3 \end{pmatrix} = \pi\, k_2 k_3.$$

2. *If $k_1 \leqslant k_2 + k_3$ then*
$$I_3 \begin{pmatrix} a_1, a_1, a_1 \\ k_1, k_2, k_3 \end{pmatrix} = \pi \left(k_2 k_3 - \frac{(k_2 + k_3 - k_1)^2}{4} \right).$$

Proof. The first part follows from Theorem 2 and the second from Corollary 1 in [4].

Alternatively, Corollary 3.6 may be derived from Corollary 3.4 on letting a tend to zero. Again, this can be automatically done in a computer algebra system such as *Mathematica 7* or *Maple 14*. □

4 Especially special cases of sinc integrals

The same phenomenon as in equation (5) and in Corollary 3.6 leads to one of the most striking examples in [4]. Consider the following example of a re-normalized I_n integral, in which we set

$$J_n := \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{2n+1}\right) \mathrm{d}x.$$

Then — as *Maple* and *Mathematica* are able to confirm — we have the following evaluations:

$$\begin{aligned}
J_0 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \,\mathrm{d}x = \pi,\\
J_1 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \mathrm{d}x = \pi,\\
&\vdots\\
J_6 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{13}\right) \mathrm{d}x = \pi.
\end{aligned}$$

As explained in detail in [4] or [5, Chapter 2], the seemingly obvious pattern — a consequence of Corollary 2.6 — is then confounded by

$$\begin{aligned}
J_7 &= \int_{-\infty}^{\infty} \operatorname{sinc} x \cdot \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) \mathrm{d}x\\
&= \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi < \pi,
\end{aligned}$$

where the fraction is approximately $0.99999999998529\ldots$ which, depending on the precision of calculation, numerically might not even be distinguished from 1.

This is a consequence of the following general evaluation given in [4]:

Theorem 4.1 (First bite)**.** *Denote* $K_m = k_0 + k_1 + \ldots + k_m$. *If* $2k_j \geqslant k_n > 0$ *for* $j = 0, 1, \ldots, n-1$ *and* $K_n > 2k_0 \geqslant K_{n-1}$ *then*

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \frac{\sin(k_j x)}{x} \,\mathrm{d}x = \pi k_1 k_2 \cdots k_n - \frac{\pi}{2^{n-1} n!}(K_n - 2k_0)^n. \tag{29}$$

Note that Theorem 4.1 is a "first-bite" extension of Corollary 2.6: assuming only that $k_j > 0$ for $j = 0, 1, \ldots, n$ then if $2k_0 > K_n$ the integral evaluates to $\pi k_1 k_2 \cdots k_n$.

Theorem 4.1 makes clear that the pattern $J_n = \pi$ for $n = 0, 1, \ldots, 6$ breaks for J_7 because

$$\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{15} > 1$$

whereas all earlier partial sums are less than 1. Geometrically, the situation is as follows: in light of (5), the integral J_n may be interpreted as the volume of the part of a hypercube lying between two planes. For $n = 7$ these planes intersect with the hypercube for the first time.

Example 4.2 (A probabilistic interpretation)**.** Let us illustrate Theorem 4.1 using the probabilistic point of view mentioned in (6) at the end of the introduction. As such the integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(k_0 x)}{x} \frac{\sin(k_1 x)}{k_1 x} \frac{\sin(k_2 x)}{k_2 x} \, \mathrm{d}x$$

is the probability that $|k_1 X_1 + k_2 X_2| \leqslant k_0$ where X_1, X_2 are independent random variables distributed uniformly on $[-1, 1]$.

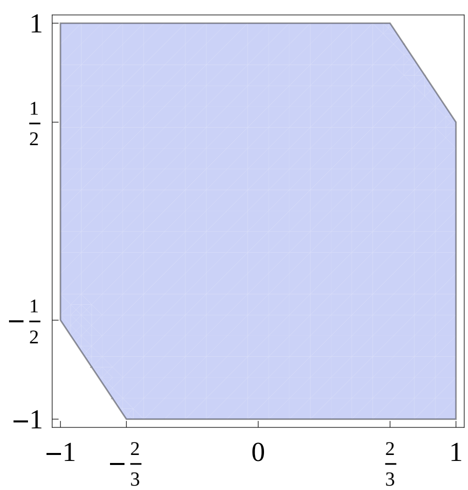

Figure 4: The event $|3X_1 + 2X_2| \leqslant 4$

In the case $k_0 = 4$, $k_1 = 3$, $k_2 = 2$, for example, this event is represented as the shaded area in Figure 4. Since each of the removed triangular corners has sides of length $1/2$ and $1/3$, this region has area $23/6$. Because the total area is 4 the probability of the event in question is $23/24$. Thus,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \frac{\sin(3x)}{3x} \frac{\sin(2x)}{2x} \, \mathrm{d}x = \frac{23}{24} = \frac{1}{3 \cdot 2} \left(3 \cdot 2 - \frac{(3 + 2 - 4)^2}{2 \cdot 2} \right)$$

in agreement with Theorem 4.1. $\lozenge$

Let us return to the example of the integrals J_n. Even past $n = 7$ we do have a surprising equality [2] of these integrals and corresponding Riemann sums. This alternative evaluation of the integrals J_n is

$$\int_{-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{x}{2j+1}\right) \mathrm{d}x = \sum_{m=-\infty}^{\infty} \prod_{j=0}^{n} \operatorname{sinc}\left(\frac{m}{2j+1}\right) \tag{30}$$

which is valid for $n = 1, 2, \ldots, 7, 8, \ldots, 40248$. The "first-bite" phenomenon is seen here again but at larger n: for $n > 40248$ this equality fails as well; the sum being strictly bigger than the integral. As in the case of (29) there is nothing special about the choice of parameters $k_j = \frac{1}{2j+1}$ in the sinc functions. Indeed, the following general result is proved in [2]:

Theorem 4.3. *Suppose that $k_1, k_2, \ldots, k_n > 0$. If $k_1 + k_2 + \ldots + k_n < 2\pi$ then*

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j x) \, \mathrm{d}x = \sum_{m=-\infty}^{\infty} \prod_{j=1}^{n} \operatorname{sinc}(k_j m). \tag{31}$$

Note that the condition $k_1 + k_2 + \ldots + k_n < 2\pi$ may always be satisfied through a common rescaling $k_j \to \tau k_j$ of the parameters k_j at the expense of writing the sinc integral as a sinc sum with differently scaled parameters.

As a consequence of Theorem 4.3, we see that (30) holds for n provided that

$$\sum_{j=0}^{n} \frac{1}{2j+1} < 2\pi$$

which is true precisely for the range of n specified above.

Remark 4.4. With this insight, it is not hard to contrive more persistent examples. An entertaining example given in [2] is taking the reciprocals of primes: using the Prime Number Theorem one estimates that the sinc integrals equal the sinc sums until the number of products is about 10^{176}. That of course makes it rather unlikely to find, by mere testing, an example where the two are unequal. Even worse for the naive tester is the fact that the discrepancy between integral and sum is always less than $10^{-10^{86}}$ (and even smaller if the Riemann hypothesis is true). $\lozenge$

A related integral which because of its varied applications has appeared repeatedly in the literature, see e.g. [13] and the references therein, is

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos(bx)\,\mathrm{d}x \tag{32}$$

which, for $0 \leqslant b < n$, has the closed form

$$\frac{1}{2^{n-1}(n-1)!}\sum_{0\leqslant k<(n+b)/2}(-1)^k\binom{n}{k}(n+b-2k)^{n-1}.$$

To give an idea of the range of applications, we only note that the authors of [13] considered the integral (32) because it is proportional to "the intermodulation distortion generated by taking the nth power of a narrow-band, high-frequency white noise"; on the other hand, the recent [1] uses (32) with $b = 0$ to obtain an improved lower bound for the Erdős-Moser problem.

If $b \geqslant n$ then the integral (32) vanishes. The case $b = 0$ in (32) is the interesting special case of I_n with $k_1 = \ldots = k_n = 1$ and $a_1, \ldots, a_n = 0$. Its evaluation appears as an exercise in [18, p. 123]; in [4] it is demonstrated how it may be derived using the present methods.

5 The case $n \geqslant 4$

Returning to Theorem 2.4 we now show — somewhat briefly — that in certain general cases the evaluation of the integral I_n may in essence be reduced to the evaluation of the integral I_m for some $m < n$. In particular, we will see that Corollary 2.6 is the most basic such case — corresponding to $m = 1$.

In order to exhibit this general structure of the integrals I_n, we introduce the notation

$$I_{n,j} := \sum_{A,B} \alpha_{j,A,B}\, C_{j,A,B} \tag{33}$$

so that, by (16), $I_n = \sum_{j=1}^n I_{n,j}$.

Theorem 5.1 (Substructure)**.** *Assume that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n > 0$, and that $a_1, a_2, \ldots, a_n$ are distinct. Suppose that there is some m such that for all $\varepsilon \in \{-1, 1\}^n$ we have*

$$\operatorname{sgn}(\varepsilon_1 k_1 + \ldots + \varepsilon_m k_m + \ldots + \varepsilon_n k_n) = \operatorname{sgn}(\varepsilon_1 k_1 + \ldots + \varepsilon_m k_m). \tag{34}$$

Then

$$I_n = \sum_{j=1}^{m} I_{m,j} \prod_{i>m} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i}. \tag{35}$$

Proof. Note that in light of (15) and (34) we have $\alpha_{j,A,B} = 0$ unless $\{m+1, \ldots, n\} \subset B$. To see this assume that there is some index $k > m$ such that $k \in A$ or $k = j$. Then the term in (15) contributed by $\varepsilon \in \{-1, 1\}^n$ has opposite sign as the term contributed by ε', where ε' is obtained from ε by flipping the sign of ε_k. The claim now follows from Theorem 2.4. □

Remark 5.2. The condition (34) may equivalently be stated as

$$\min |\varepsilon_1 k_1 + \ldots + \varepsilon_m k_m| > k_{m+1} + \ldots + k_n \tag{36}$$

where the minimum is taken over $\varepsilon \in \{-1, 1\}^m$. We idly remark that, for large m, computing this minimum is a hard problem. In fact, in the special case of integral k_j just deciding whether the minimum is zero (which is equivalent to the *partition problem* of deciding whether the parameters k_j can be partitioned into two sets with the same sum) is well-known to be NP-complete [8, Section 3.1.5]. ♢

Observe that the case $m = 1$ of Theorem 5.1 together with the basic evaluation (4) immediately implies Corollary 2.6. This is because the condition (34) holds for $m = 1$ precisely if $k_1 > k_2 + \ldots + k_n$.

If (34) holds for $m = 2$ then it actually holds for $m = 1$ provided that the assumed inequality $k_1 \geqslant k_2$ is strict. Therefore the next interesting case is $m = 3$. The final evaluation makes this case explicit. It follows from Corollary 3.2.

Corollary 5.3 (A second n-dimensional case)**.** *Let $n \geqslant 3$. Assume that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n > 0$, and that $a_1, a_2, \ldots, a_n$ are distinct. If*

$$k_1 \leqslant k_2 + \ldots + k_n \quad \text{and} \quad k_2 + k_3 - k_1 \geqslant k_4 + \ldots + k_n$$

then

$$I_n = \frac{\pi}{2} \sum_{j=1}^{3} \prod_{i \geqslant 4} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i} \left[\prod_{i \leqslant 3, i \neq j} \frac{\sin(k_i(a_j - a_i))}{a_j - a_i} + \prod_{i \leqslant 3, i \neq j} \frac{\cos(k_i(a_j - a_i))}{a_j - a_i} \right].$$

The cases $m \geqslant 4$ quickly become much more involved. In particular, the condition (34) becomes a set of inequalities. To close, we illustrate with the first case not covered by Corollaries 2.6 and 5.3:

Example 5.4. As usual, assume that $k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4 > 0$, and that a_1, a_2, a_3, a_4 are distinct. If $k_1 < k_2 + k_3 + k_4$ (hence Corollary 2.6 does not apply) and $k_1 + k_4 > k_2 + k_3$ (hence Corollary 5.3 does not apply either) then

$$I_4 = \frac{\pi}{4} \sum_{j=1}^{4} \sum_{A,B} C_{j,A,B} + \frac{\pi}{2} \prod_{i \neq 1} \frac{\sin(k_i(a_1 - a_i))}{a_1 - a_i} - \frac{\pi}{2} \sum_{j=2}^{4} \frac{\sin(k_1(a_j - a_1))}{a_j - a_1} \prod_{i \neq 1,j} \frac{\cos(k_i(a_j - a_i))}{a_j - a_i} \tag{37}$$

where the summation in the first sum is as in Theorem 2.4. Note that the terms $I_{4,j}$ of (33) are implicit in (37) and may be used to make the case $m = 4$ of Theorem 5.1 explicit as has been done in Corollary 5.3 for $m = 3$. $\diamondsuit$

6 Conclusions

We present these results for several reasons. First is the intrinsic beauty and utility of the sinc function. It is important in so many areas of computing, approximation theory, and numerical analysis. It is used in interpolation and approximation of functions, approximate evaluation of transforms — e.g., Hilbert, Fourier, Laplace, Hankel, and Mellin transforms as well as the fast Fourier transform — see [16]. It is used in approximating solutions of differential and integral equations, in image processing [7], in other signal processing and in information theory. It is the Fourier transform of the box filter and so central to the understanding of the Gibbs phenomenon [17] and its descendants. Much of this is nicely described in [9].

Second is that the forensic nature of the mathematics was entertaining. It also made us reflect on how computer packages and databases have changed mathematics over the past forty to fifty years. As our hunt for the history of this MONTHLY problem indicates, better tools for searching our literature are much needed. Finally some of the evaluations merit being better known as they are excellent tests of both computer algebra and numerical integration.

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