# Expectations of Random Walks

Armin Straub, James Wan

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# The random walk integrals

## Definition

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

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- PRIMA, quantum chemistry, code analysis in WWII.

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- Values:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

## $W_n(k)$ at even integers $k$

- Easier (no square root), and combinatorial:

| $k$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| $W_2(k)$ | 1 | 2 | 6 | 20 | 70 | 252 |
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# $W_n(k)$ at even integers $k$ (continued)

- General formula:

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- Recursions by Sister Celine, eg.:

$$\begin{aligned}(k+2)^2 W_3(2k+4) - (10k^2+30k+23) W_3(2k+2) \\ +9(k+1)^2 W_3(2k) &= 0.\end{aligned}$$

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- Some formulas from B-B-B-G:

$$\begin{aligned}\left(\sum_{k\geqslant 0} W_3(2k)(-x)^k\right)^2 &= \sum_{k\geqslant 0} W_2(2k)^3 \frac{x^{3k}}{((1+x)^3(1+9x))^{k+1/2}} \\ &= \sum_{k\geqslant 0} W_2(2k)W_3(2k)\frac{(-x(1+x)(1+9x))^k}{((1-3x)(1+3x))^{2k+1}} \\ &= \sum_{k\geqslant 0} W_4(2k)\frac{x^k}{((1+x)(1+9x))^{k+1}}.\end{aligned}$$

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- What about $W_5, W_6, \ldots$?

# Binomial expansion of $W_n(s)$

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$$W_n(s) = n^s \sum_{m \geqslant 0} \frac{(-1)^m}{n^{2m}} \binom{\frac{s}{2}}{m} \underbrace{\int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \mathrm{d}\boldsymbol{x}}_{=:I_{n,m}}$$

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- Experimentally found recursion for $I_{3,m}$.

# $W_n(s)$ as a sum

- Looking up $I_{3,m}$ on Sloane: get **A093388**

$$1, 6, 42, 312, 2394, 18756, 149136, \ldots$$

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$$\begin{aligned} & (8xyz - (x+y)(y+z)(z+x))^m \\ = \; & \left(3^2 xyz - (x+y+z)(xy+yz+zx)\right)^m \end{aligned}$$

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$$\left(n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n)\right)^m$$

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- Leads to conjecture

$$W_n(s) = n^s \sum_{m \geqslant 0} (-1)^m \binom{\frac{s}{2}}{m} \sum_{k=0}^{m} \frac{(-1)^k}{n^{2k}} \binom{m}{k} \sum_{\sum a_i = k} \binom{k}{a_1, \ldots, a_n}^2.$$

# $W_n(s)$ as a sum (continued)

- In particular,

$$W_3 = 3\sum_{n=0}^{\infty}\binom{1/2}{n}\left(-\frac{8}{9}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{8}\right)^k\sum_{j=0}^{k}\binom{k}{j}^3.$$

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- The elliptic integrals arise out of $W_2$:

$$\frac{W_2(1)}{2} = \sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(1-2k)2^{4k}} = \frac{2\mathrm{E}(1)}{\pi} = {}_2F_1\left(\begin{matrix}1/2, -1/2\\ 1\end{matrix}\middle|1\right).$$

# $W_n(s)$ as a sum (idea of proof)

- Need $I_{n,m} = \int_{[0,1]^n} \left( 4 \sum_{i<j} \sin^2(\pi(x_j - x_i)) \right)^m \mathrm{d}\boldsymbol{x}$ as const. term of

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$$\begin{aligned} &\left(n^2 - (x_1 + \ldots + x_n)(1/x_1 + \ldots + 1/x_n)\right)^m \\ = &\left(\sum_{i<j}\left(2 - \frac{x_i}{x_j} - \frac{x_j}{x_i}\right)\right)^m \\ = &\left(-\sum_{i<j}\frac{(x_j - x_i)^2}{x_i x_j}\right)^m \end{aligned}$$

- Now, expand the $m$-th power on both sides...

# Recursions for $W_n(s)$

- Have seen: for integers $k$

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**Theorem (Carlson's Theorem)**

*If $f(z)$ is analytic for $\mathrm{Re}(z) \geq 0$,* "nice"*, and*

$$f(0)=0, f(1)=0, f(2)=0, \ldots$$

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- $W_n(s)$ nice!

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- We have

$$W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = \underbrace{{}_3F_2\left(\left.\begin{matrix} 1/2, -k, -k \\ 1, 1 \end{matrix}\right| 4\right)}_{=:V_3(2k)}$$

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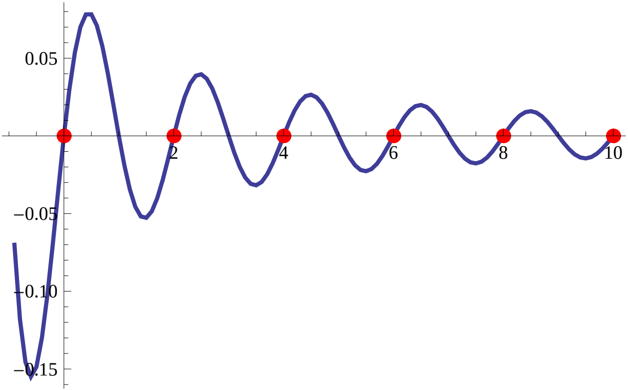
## Conjecture (Dirk Nuyens)

*For integers $k$,*

$$W_3(k) \stackrel{?}{=} \operatorname{Re} V_3(k).$$

# $W_3$ versus $V_3$

- Still, $V_3(s) = {}_3F_2\left(\begin{matrix}1/2, -s/2, -s/2\\ 1, 1\end{matrix}\middle| 4\right)$.
- Here's $\operatorname{Re}(W_3(s) - V_3(s))$:



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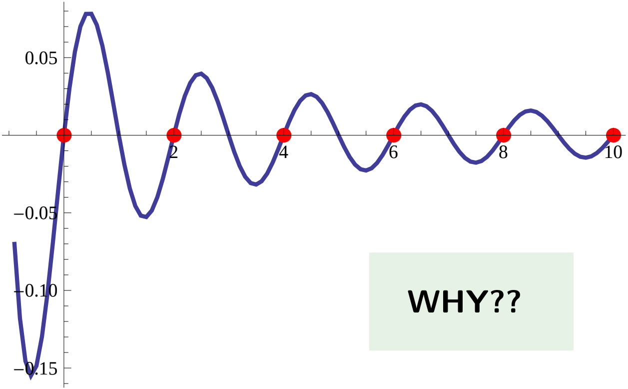


**WHY??**

# $W_3$ versus $V_3$ (part 2)

- Why is $W_3(s) \neq V_3(s)$ for non-integers $s$?
- Both functions are analytic for $\mathrm{Re}(s) \geq 0$ and satisfy the same recursion.

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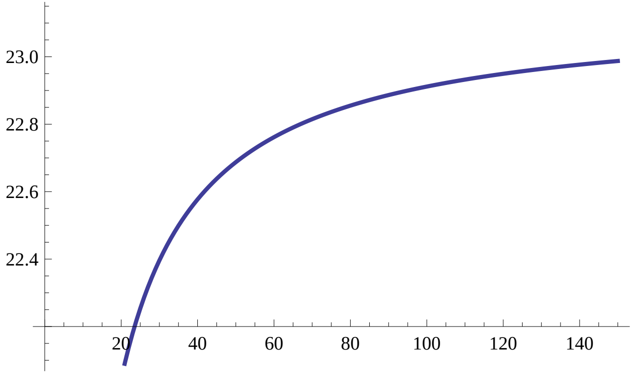

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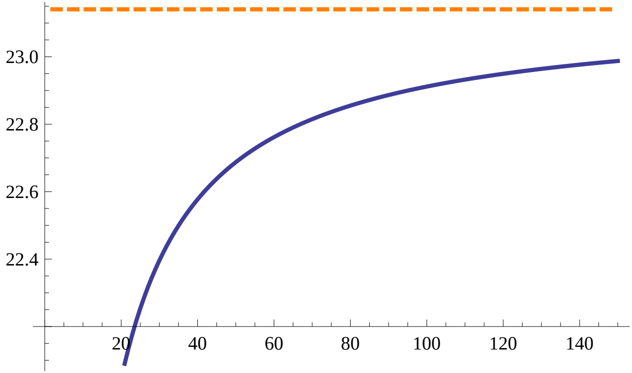

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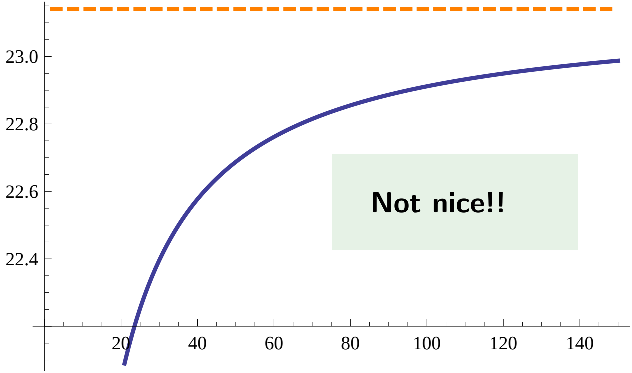

# $W_3(s)$ at integers (continued)

- Based on Dirk's conjecture,

$$\begin{aligned} W_3(1) &\overset{?}{=} \frac{4\sqrt{3}}{3}\left({}_3F_2\left(\left.\begin{matrix}-1/2, -1/2, -1/2\\ 1, 1\end{matrix}\right|\frac{1}{4}\right) - \frac{1}{\pi}\right) \\ &+ \frac{\sqrt{3}}{24}\,{}_3F_2\left(\left.\begin{matrix}1/2, 1/2, 1/2\\ 2, 2\end{matrix}\right|\frac{1}{4}\right). \end{aligned}$$

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- Equivalently, for $k_3 = \frac{\sqrt{3}-1}{2\sqrt{2}}$,

$$W_3(1) \overset{?}{=} 2\sqrt{3}\frac{K^2(k_3)}{\pi^2} + \sqrt{3}\frac{1}{K^2(k_3)} = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right).$$

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- Also $W_3(3), W_3(5), \ldots$

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- We have

$$W_4(2k) = \sum_{a_1+\cdots+a_4=k} \binom{k}{a_1,\ldots,a_4}^2 = \underbrace{\sum_{j\geq 0} \binom{k}{j}^2 {}_3F_2\left(\left.\begin{matrix}1/2, -k+j, -k+j \\ 1, 1\end{matrix}\right| 4\right)}_{=: V_4(2k)}.$$

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## Conjecture

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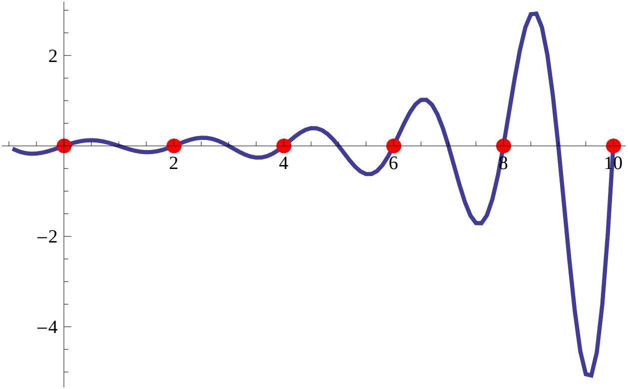

# $W_4(s)$ at integers (continued)

- Based on the previous conjecture,

$$\begin{aligned} W_4(1) \approx\ & 1.79909247984285103353260284584610891006 \\ & 6282003291620456626641773598854266932120 \\ & 5752411619305734748280560170144445 1798\ldots \end{aligned}$$

in agreement with the $8$ digits obtained by quadrature.

# $W_4(s)$ at integers (continued)

- Based on the previous conjecture,

$$W_4(1) \approx 1.79909247984285103353260284584610891006\\62820032916204566266417735988542669321 20\\5752411619305734748280560170144445179 8\ldots$$

in agreement with the $8$ digits obtained by quadrature.

- $W_4(1)$ a sum of four ${}_4F_3$'s?? Any other "closed" form?

# Bold finale

- For some (even?) $n$ and some (integer?) $s$,

$$W_n(s) \overset{??}{=} \sum_{j\geq 0} \binom{s/2}{j}^2 W_{n-1}(s-2j).$$

# Bold finale

- For some (even?) $n$ and some (integer?) $s$,

$$W_n(s) \overset{??}{=} \sum_{j\geq 0} \binom{s/2}{j}^2 W_{n-1}(s-2j).$$

- True for $n=2$, true for even $s$.
- Probably true for $n=4$ and integer $s$.

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**THANK YOU!**