Random walks in the plane

Armin Straub

Tulane University, New Orleans

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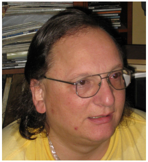

Joint work with: Jon Borwein (U. of Newcastle, AU), Dirk Nuyens (K.U.Leuven, BE), James Wan (U. of Newcastle, AU)

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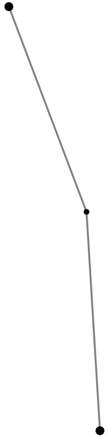

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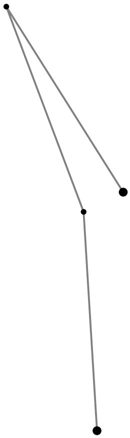

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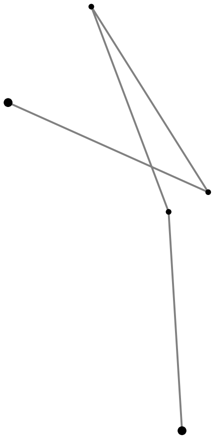

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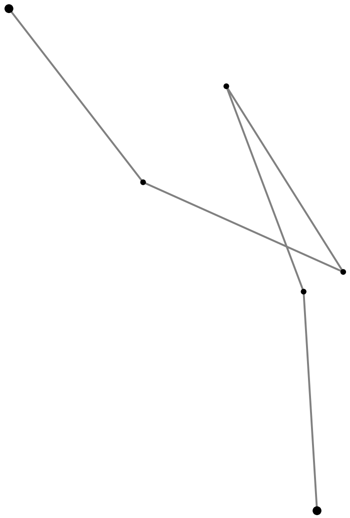
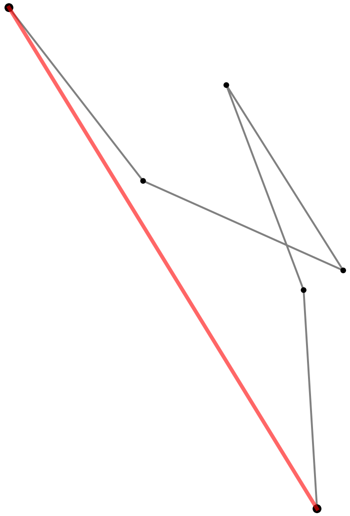


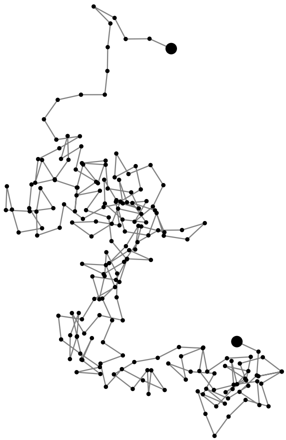
Random walks in the plane

- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.
- We are interested in the distance traveled in n steps.

  For instance, how large is this distance on average?



Long walks



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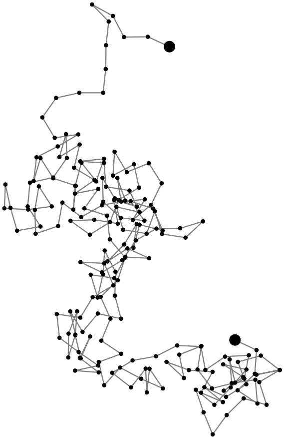
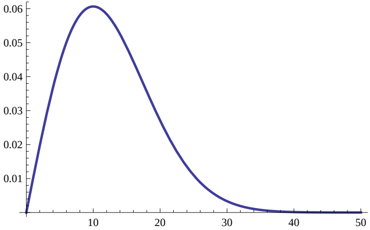


- Asked by Karl Pearson in Nature in 1905
- For long walks, the probability density is approximately $\dfrac{2x}{n} e^{-x^2/n}$
- For instance, for $n = 200$:



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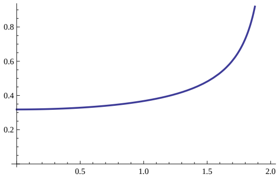
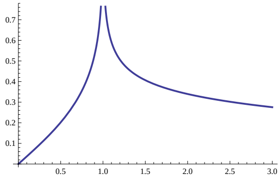
Lord Rayleigh. "The problem of the random walk." *Nature*, **72**, 1905.

Densities

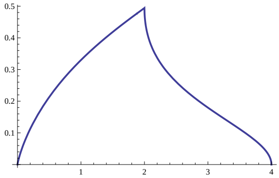
$n = 2$



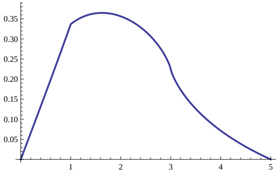
$n = 3$



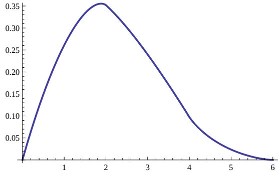
$n = 4$



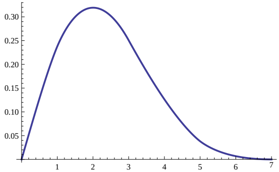
$n = 5$



$n = 6$



$n = 7$



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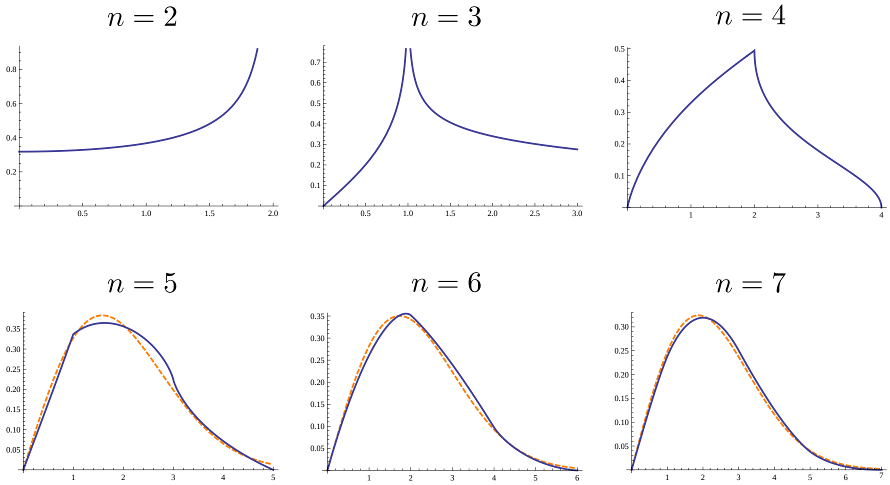

Moments

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- Represent the kth step by the complex number $e^{2\pi i x_k}$.
  The sth moment of the distance after n steps is:

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

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  In particular, $W_n(1)$ is the average distance after n steps.
- This is hard to evaluate numerically to high precision. For instance, Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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Even moments

n	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
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$$W_5(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2(k-j)}{k-j} \sum_{\ell=0}^{j} \binom{j}{\ell}^2 \binom{2\ell}{\ell}$$

Combinatorics

Theorem (Borwein-Nuyens-S-Wan)

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- $f_n(k)$ satisfies recurrences and convolutions.

L. B. Richmond and J. Shallit. "Counting abelian squares." *The Electronic Journal of Combinatorics*, **16**, 2009.

P. Barrucand. "Sur la somme des puissances des coefficients multinomiaux et les puissances successives d'une fonction de Bessel." *C. R. Acad. Sci. Paris*, **258**, 5318–5320, 1964.

Functional Equations for $W_n(s)$

- For integers $k \geqslant 0$,

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If $f(z)$ is analytic for $\operatorname{Re}(z) \geqslant 0$, "nice", and

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- So we get complex functional equations like

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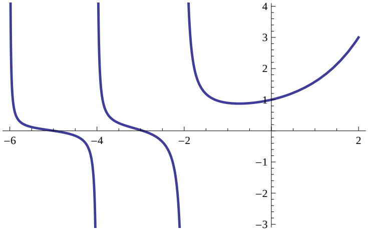
- This gives analytic continuations of $W_n(s)$ to the complex plane, with poles at certain negative integers.



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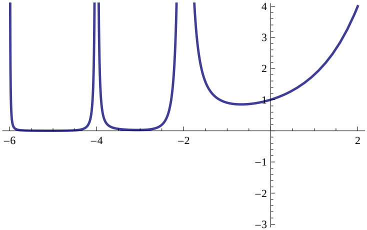

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- Again, from combinatorics:

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = \underbrace{{}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, -k, -k \\ 1, 1 \end{matrix} \right| 4 \right)}_{=: V_3(2k)}$$

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Theorem (Borwein-Nuyens-S-Wan)

For integers k we have $W_3(k) = \operatorname{Re}\, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, -\frac{k}{2}, -\frac{k}{2} \\ 1, 1 \end{matrix} \right| 4 \right)$.

$W_3(1) = 1.57459723755189\ldots = ?$

Corollary (Borwein-Nuyens-S-Wan)

$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

- Similar formulas for $W_3(3), W_3(5), \ldots$

A generating function

- Recall:

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

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$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

- Therefore:

$$\sum_{k=0}^{\infty} W_n(2k) \frac{(-x)^k}{(k!)^2} = \sum_{k=0}^{\infty} \sum_{a_1+\cdots+a_n=k} \frac{(-x)^k}{(a_1!)^2\cdots(a_n!)^2}$$

$$= \left(\sum_{a=0}^{\infty} \frac{(-x)^a}{(a!)^2} \right)^n = J_0(2\sqrt{x})^n$$

Ramanujan's Master Theorem

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For "nice" analytic functions φ*,*

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- Begs to be applied to

$$\sum_{k=0}^{\infty} W_n(2k) \frac{(-x)^k}{(k!)^2} = J_0(2\sqrt{x})^n$$

by setting $\varphi(k) = \dfrac{W_n(2k)}{k!}$

Ramanujan's Master Theorem

- We find:

$$W_n(-s) = 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\int_0^\infty x^{s-1}J_0^n(x)\,\mathrm{d}x$$

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- First and more inspiredly found by David Broadhurst
  building on work of J.C. Kluyver

David Broadhurst. "Bessel moments, random walks and Calabi-Yau equations." Preprint, Nov 2009.

J.C. Kluyver. "A local probability problem." *Nederl. Acad. Wetensch. Proc.*, **8**, 341–350, 1906.

A convolution formula

Conjecture

For even n,

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- True for even s
- True for $n = 2$
- Now proven up to some technical growth conditions

You will have to look at the papers to find...

- a hyper-closed form for $W_4(1)$,
- Meijer-G and hypergeometric expressions for $W_3(s)$ and $W_4(s)$,
- evaluations of derivatives including

$$W_3'(0) = \frac{1}{\pi}\,\mathrm{Cl}\left(\frac{\pi}{3}\right), \quad W_4'(0) = \frac{7\zeta(3)}{2\pi^2},$$

- expressions for residues at the poles of $W_n(s)$,
- ...

References

J. Borwein, D. Nuyens, A. Straub, and J. Wan. "Random Walk Integrals." Preprint, Oct 2009.

J. Borwein, A. Straub, and J. Wan. "Three-Step and Four-Step Random Walk Integrals." Preprint, May 2010.

Both preprints as well as this talk are/will be available from:
`http://arminstraub.com`

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