On the method of brackets

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Joint work with: Ivan Gonzalez, Universidad Santa Maria, Chile; Victor Moll, Tulane University, U.S.

Advertisement

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- Also:

$$\int_0^\infty \int_0^\infty \int_0^\infty x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\mathrm{e}^{x_1 m^2} \mathrm{e}^{-\frac{x_1 x_2}{x_1+x_2+x_3} s}}{(x_1+x_2+x_3)^{D/2}}\,\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

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$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^{k+1}} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n}$$

Origin and more advertising

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- Alternative to the practice of introducing Mellin-Barnes representations for everything
- The method of brackets ...
  - is distilled to a set of 3 simple rules
  - is applicable to a wide class of definite integrals
  - is comfortably applied by hand for many simpler integrals
  - is (quite) automatable
    Karen Kohl is working on an implementation in SAGE.

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The *gamma function*

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Evaluating bracket series

Rule

$$\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*),$$

where n^* is the solution of the equation $an + b = 0$.

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$$= \frac{1}{2} a^{n^*} \Gamma(-n^*) = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

More interesting example

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$$= \frac{1}{\alpha} \sum_k (-1)^k \binom{2k}{k} \left(\frac{\beta}{2\alpha}\right)^{2k} = \frac{1}{\sqrt{\alpha^2+\beta^2}}$$

with the series converging for $\beta < \alpha$.

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  with the series converging for $\beta < \alpha$.
- Similarly, for m free:

$$\frac{1}{2} \sum_m \phi_m \alpha^m \left(\frac{\beta}{2}\right)^{-m-1} \frac{\Gamma(1/2+m/2)}{\Gamma(1/2-m/2)} = \ldots = \frac{1}{\sqrt{\alpha^2+\beta^2}}$$

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$$\sum_{\{n\}} \phi_{\{n\}} f(n_1,\ldots,n_r) \langle a_{11}n_1+\cdots a_{1r}n_r+b_1 \rangle \cdots \langle a_{r1}n_1+\cdots a_{rr}n_r+b_r \rangle$$
$$= \frac{1}{|\det(A)|} f(n_1^*,\ldots,n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*),$$

where $A=(a_{ij})$ *and* (n_i^*) *such that the brackets vanish.*

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Rule (Combining)

If there are more summation indices than brackets, free variables are chosen. Each choice produces a series. Those converging in a common region are added.

Multinomial expansions

Rule (Multinomial)

$$\frac{1}{(a_1 + a_2 + \cdots + a_r)^s} = \sum_{m_1,\ldots,m_r} \phi_{\{m\}} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle s + m_1 + \cdots + m_r \rangle}{\Gamma(s)}$$

where $\phi_{\{m\}} := \phi_{m_1} \cdots \phi_{m_r}$.

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- Follows from the integral representation of $\Gamma(s)$:

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A two-dimensional example

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&= \sum_{j,n,m} \phi_{j,n,m} \frac{1}{\Gamma(-\alpha j)} \langle n+m-\alpha j\rangle \langle n+s\rangle \langle m+t\rangle
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$$= \frac{1}{\alpha} \frac{1}{\Gamma(-\alpha j^*)} \Gamma(-n^*) \Gamma(-m^*) \Gamma(-j^*) = \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \Gamma\left(\frac{s+t}{\alpha}\right)$$

A two-dimensional example

$$\int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \exp\left(-(x+y)^\alpha\right) \, \mathrm{d}x \, \mathrm{d}y$$

$$= \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} (x+y)^{\alpha j} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \sum_j \phi_j \int_0^\infty \int_0^\infty x^{s-1} y^{t-1} \sum_{n,m} \phi_{n,m} x^n y^m \frac{\langle n+m-\alpha j \rangle}{\Gamma(-\alpha j)} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \sum_{j,n,m} \phi_{j,n,m} \frac{1}{\Gamma(-\alpha j)} \langle n+m-\alpha j \rangle \langle n+s \rangle \langle m+t \rangle$$

$$n^* = -s, \quad m^* = -t, \quad j^* = -\tfrac{s+t}{\alpha}, \quad |\det| = \alpha$$

$$= \frac{1}{\alpha} \frac{1}{\Gamma(-\alpha j^*)} \Gamma(-n^*) \Gamma(-m^*) \Gamma(-j^*) = \frac{1}{\alpha} \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \Gamma\left(\frac{s+t}{\alpha}\right)$$

- *Mathematica* 7 cannot evaluate this integral.

More dimensions

- This generalizes to arbitrary dimensions:

Theorem

$$\int_0^\infty \cdots \int_0^\infty \exp\left(-(x_1 + \ldots + x_n)^\alpha\right) \prod_{i=1}^n x_i^{s_i-1} \, \mathrm{d}x_i$$
$$= \frac{1}{\alpha} \frac{\prod_{i=1}^n \Gamma(s_i)}{\Gamma(s_1 + \ldots + s_n)} \Gamma\left(\frac{s_1 + \ldots + s_n}{\alpha}\right)$$

A Bessel integral

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x$$

A Bessel integral

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \,\mathrm{d}x$$

$$= \sum_k \phi_k \int_0^\infty \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \frac{x^{2k+s-1}}{(1+x^2)^\lambda} \,\mathrm{d}x$$

A Bessel integral

$$\begin{aligned}
&\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x \\
&= \sum_k \phi_k \int_0^\infty \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \frac{x^{2k+s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x \\
&= \frac{1}{\Gamma(\lambda)} \sum_{k,n,m} \phi_{k,n,m} \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \langle n+m+\lambda \rangle \langle 2m+2k+s \rangle
\end{aligned}$$

A Bessel integral

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \,\mathrm{d}x$$
$$= \sum_k \phi_k \int_0^\infty \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \frac{x^{2k+s-1}}{(1+x^2)^\lambda} \,\mathrm{d}x$$
$$= \frac{1}{\Gamma(\lambda)} \sum_{k,n,m} \phi_{k,n,m} \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \langle n+m+\lambda \rangle \langle 2m+2k+s \rangle$$

- 3 indices, 2 brackets: 1 free variable

A Bessel integral — k free

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x$$
$$= \frac{1}{\Gamma(\lambda)} \sum_{k,n,m} \phi_{k,n,m} \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \langle n+m+\lambda \rangle \langle 2m+2k+s \rangle$$

- k free: $m^* = -k - \frac{s}{2}$ and $n^* = -\lambda + k + \frac{s}{2}$. $|\det| = 2$

A Bessel integral — k free

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- k free: $m^* = -k - \frac{s}{2}$ and $n^* = -\lambda + k + \frac{s}{2}$. $|\det| = 2$

$$\frac{1}{2\Gamma(\lambda)} \sum_k \phi_k \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \Gamma(-n^*)\Gamma(-m^*)$$

A Bessel integral — k free

$$\int_0^\infty J_0(\alpha x)\frac{x^{s-1}}{(1+x^2)^\lambda}\,\mathrm{d}x$$
$$=\frac{1}{\Gamma(\lambda)}\sum_{k,n,m}\phi_{k,n,m}\left(\frac{\alpha}{2}\right)^{2k}\frac{1}{\Gamma(k+1)}\langle n+m+\lambda\rangle\langle 2m+2k+s\rangle$$

- k free: $m^* = -k-\frac{s}{2}$ and $n^* = -\lambda+k+\frac{s}{2}$. $|\det| = 2$

$$\frac{1}{2\Gamma(\lambda)}\sum_k\phi_k\left(\frac{\alpha}{2}\right)^{2k}\frac{1}{\Gamma(k+1)}\Gamma(-n^*)\Gamma(-m^*)$$
$$=\frac{1}{2\Gamma(\lambda)}\sum_k\frac{(-1)^k}{(k!)^2}\left(\frac{\alpha}{2}\right)^{2k}\Gamma(\lambda-k-\tfrac{s}{2})\Gamma(k+\tfrac{s}{2})$$

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$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \mathrm{d}x$$
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$$= \frac{1}{2\Gamma(\lambda)} \sum_k \frac{(-1)^k}{(k!)^2} \left(\frac{\alpha}{2}\right)^{2k} \Gamma(\lambda - k - \tfrac{s}{2})\Gamma(k + \tfrac{s}{2})$$
$$= \frac{\Gamma(\frac{s}{2})\Gamma(\lambda - \frac{s}{2})}{2\Gamma(\lambda)} {}_1F_2 \left(\left. \begin{matrix} \frac{s}{2} \\ 1, 1-\lambda+\frac{s}{2} \end{matrix} \right| \frac{\alpha^2}{4} \right)$$

A Bessel integral — the contributions

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x$$
$$= \frac{1}{\Gamma(\lambda)} \sum_{k,n,m} \phi_{k,n,m} \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{\Gamma(k+1)} \langle n+m+\lambda \rangle \langle 2m+2k+s \rangle$$

- k free: $\frac{\Gamma(\frac{s}{2})\Gamma(\lambda-\frac{s}{2})}{2\Gamma(\lambda)} {}_1F_2\left(\left.\begin{matrix} \frac{s}{2} \\ 1, 1-\lambda+\frac{s}{2} \end{matrix}\right| \frac{\alpha^2}{4}\right)$

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- n free: $\left(\frac{\alpha}{2}\right)^{2\lambda-s} \frac{\Gamma(-\lambda+\frac{s}{2})}{2\Gamma(\lambda+1-\frac{s}{2})} {}_1F_2 \left(\left. \begin{matrix} \lambda \\ 1+\lambda-\frac{s}{2}, 1+\lambda-\frac{s}{2} \end{matrix} \right| \frac{\alpha^2}{4} \right)$

A Bessel integral — the contributions

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x$$
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- m free: $\frac{1}{2\Gamma(\lambda)} \sum_m \frac{(-1)^m}{m!} \left(\frac{\alpha}{2}\right)^{-2m-s} \frac{\Gamma(m+\lambda)\Gamma(m+\frac{s}{2})}{\Gamma(1-m-\frac{s}{2})}$

  This series diverges.

A Bessel integral — harvesting

Theorem

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \,\mathrm{d}x = \frac{\Gamma(\frac{s}{2})\Gamma(\lambda - \frac{s}{2})}{2\Gamma(\lambda)} {}_1F_2\left(\begin{matrix} \frac{s}{2} \\ 1, 1-\lambda+\frac{s}{2} \end{matrix} \middle| \frac{\alpha^2}{4}\right) + \left(\frac{\alpha}{2}\right)^{2\lambda - s} \frac{\Gamma(-\lambda+\frac{s}{2})}{2\Gamma(\lambda+1-\frac{s}{2})} {}_1F_2\left(\begin{matrix} \lambda \\ 1+\lambda-\frac{s}{2}, 1+\lambda-\frac{s}{2} \end{matrix} \middle| \frac{\alpha^2}{4}\right)$$

A Bessel integral — harvesting

Theorem

$$\int_0^\infty J_0(\alpha x)\frac{x^{s-1}}{(1+x^2)^\lambda}\,\mathrm{d}x = \frac{\Gamma(\frac{s}{2})\Gamma(\lambda-\frac{s}{2})}{2\Gamma(\lambda)}{}_1F_2\left(\begin{matrix}\frac{s}{2}\\ 1,1-\lambda+\frac{s}{2}\end{matrix}\middle|\frac{\alpha^2}{4}\right) + \left(\frac{\alpha}{2}\right)^{2\lambda-s}\frac{\Gamma(-\lambda+\frac{s}{2})}{2\Gamma(\lambda+1-\frac{s}{2})}{}_1F_2\left(\begin{matrix}\lambda\\ 1+\lambda-\frac{s}{2},1+\lambda-\frac{s}{2}\end{matrix}\middle|\frac{\alpha^2}{4}\right)$$

Corollary ($s=2$)

$$\int_0^\infty J_0(\alpha x)\frac{x}{(1+x^2)^{\lambda+1}}\,\mathrm{d}x = \left(\frac{\alpha}{2}\right)^\lambda\frac{K_\lambda(\alpha)}{\lambda!}$$

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Theorem

$$\int_0^\infty J_0(\alpha x) \frac{x^{s-1}}{(1+x^2)^\lambda} \, \mathrm{d}x = \frac{\Gamma(\frac{s}{2})\Gamma(\lambda - \frac{s}{2})}{2\Gamma(\lambda)} {}_1F_2\left(\begin{matrix} \frac{s}{2} \\ 1, 1-\lambda+\frac{s}{2} \end{matrix} \middle| \frac{\alpha^2}{4} \right)$$
$$+ \left(\frac{\alpha}{2}\right)^{2\lambda - s} \frac{\Gamma(-\lambda + \frac{s}{2})}{2\Gamma(\lambda + 1 - \frac{s}{2})} {}_1F_2\left(\begin{matrix} \lambda \\ 1+\lambda-\frac{s}{2}, 1+\lambda-\frac{s}{2} \end{matrix} \middle| \frac{\alpha^2}{4} \right)$$

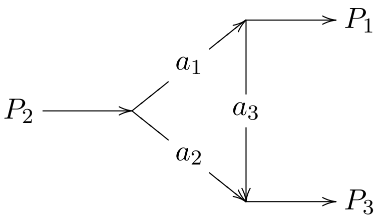
Corollary ($s = 2$)

$$\int_0^\infty J_0(\alpha x) \frac{x}{(1+x^2)^{\lambda+1}} \, \mathrm{d}x = \left(\frac{\alpha}{2}\right)^\lambda \frac{K_\lambda(\alpha)}{\lambda!}$$

Corollary

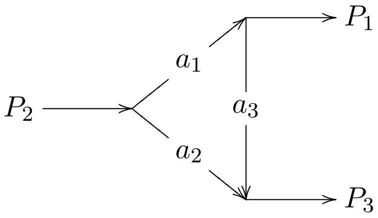
$$\int_0^\infty J_0(\alpha x) \frac{x}{(1+x^2)^{3/2}} \, \mathrm{d}x = \mathrm{e}^{-\alpha}$$

A Feynman diagram



- Propagator associated to the index a_1 has mass m
- $P_1^2 = P_3^2 = 0$ and $P_2^2 = (P_1 + P_3)^2 = s$

A Feynman diagram



- Propagator associated to the index a_1 has mass m
- $P_1^2 = P_3^2 = 0$ and $P_2^2 = (P_1 + P_3)^2 = s$
- D-dimensional representation in Minkowski space is given by

$$G = \int \frac{\mathrm{d}^D q}{i\pi^{D/2}} \frac{1}{[(P_1+q)^2 - m^2]^{a_1} [(P_3-q)^2]^{a_2} [q^2]^{a_3}}.$$

E. E. Boos and A. I. Davydychev. "A method for evaluating massive Feynman integrals." *Jour. Phys. A*, **41**, 1991.

The associated Feynman integral

- Schwinger parametrization leads to $G = \frac{(-1)^{-D/2}}{\prod_{j=1}^{3} \Gamma(a_j)} H$ with

$$H := \int_0^\infty \int_0^\infty \int_0^\infty x_1^{a_1-1} x_2^{a_2-1} x_3^{a_3-1} \frac{\mathrm{e}^{x_1 m^2} \mathrm{e}^{-\frac{x_1 x_2}{x_1+x_2+x_3} s}}{(x_1+x_2+x_3)^{D/2}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3.$$

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- First

$$e^{x_1 m^2} e^{-\frac{x_1 x_2}{x_1+x_2+x_3} s} = \sum_{n_1,n_2} \phi_{\{n\}} (-1)^{n_1} m^{2n_1} s^{n_2} \frac{x_1^{n_1+n_2} x_2^{n_2}}{(x_1+x_2+x_3)^{n_2}}$$

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$$\mathrm{e}^{x_1 m^2} \mathrm{e}^{-\frac{x_1 x_2}{x_1+x_2+x_3} s} = \sum_{n_1, n_2} \phi_{\{n\}} (-1)^{n_1} m^{2n_1} s^{n_2} \frac{x_1^{n_1+n_2} x_2^{n_2}}{(x_1+x_2+x_3)^{n_2}}$$

- Then expand

$$\frac{1}{(x_1+x_2+x_3)^{D/2+n_2}} = \sum_{n_3, n_4, n_5} \phi_{\{n\}} x_1^{n_3} x_2^{n_4} x_3^{n_5} \frac{\left\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \right\rangle}{\Gamma(\frac{D}{2} + n_2)}$$

Evaluation

- The resulting bracket series is

$$H = \sum_{\{n\}} \phi_{\{n\}} (-m^2)^{n_1} s^{n_2} \frac{\left\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \right\rangle}{\Gamma(\frac{D}{2} + n_2)}$$
$$\times \ \langle a_1 + n_1 + n_2 + n_3 \rangle \langle a_2 + n_2 + n_4 \rangle \langle a_3 + n_5 \rangle \, .$$

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- Possible choices for free variables are n_1, n_2, and n_4.

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$$H = \sum_{\{n\}} \phi_{\{n\}} (-m^2)^{n_1} s^{n_2} \frac{\left\langle \frac{D}{2} + n_2 + n_3 + n_4 + n_5 \right\rangle}{\Gamma(\frac{D}{2} + n_2)} \times \langle a_1 + n_1 + n_2 + n_3 \rangle \langle a_2 + n_2 + n_4 \rangle \langle a_3 + n_5 \rangle .$$

- Possible choices for free variables are n_1, n_2, and n_4.
- The series associated to n_2 converges for $|\frac{s}{m^2}| < 1$:

Theorem

$$H = \eta_2 \cdot {}_2F_1 \left(\left. \begin{matrix} a_1 + a_2 + a_3 - \frac{D}{2}, a_2 \\ \frac{D}{2} \end{matrix} \right| \frac{s}{m^2} \right)$$

with η_2 defined by

$$\eta_2 = (-m^2)^{\frac{D}{2} - a_1 - a_2 - a_3} \frac{\Gamma(a_2)\Gamma(a_3)\Gamma\left(\frac{D}{2} - a_2 - a_3\right)\Gamma\left(a_1 + a_2 + a_3 - \frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)}.$$

Evaluation II

- Similarly, the series associated to n_1, n_4 converges for $\left|\frac{m^2}{s}\right| < 1$:

Theorem

$$H = \eta_1 \cdot {}_2F_1\left(\left. \begin{matrix} a_1 + a_2 + a_3 - \frac{D}{2}, 1 + a_1 + a_2 + a_3 - D \\ 1 + a_1 + a_3 - \frac{D}{2} \end{matrix} \right| \frac{m^2}{s} \right) + \eta_4 \cdot {}_2F_1\left(\left. \begin{matrix} 1 + a_2 - \frac{D}{2}, a_2 \\ 1 - a_1 - a_3 + \frac{D}{2} \end{matrix} \right| \frac{m^2}{s} \right)$$

with η_1, η_4 *defined by*

$$\eta_1 = s^{\frac{D}{2} - a_1 - a_2 - a_3} \frac{\Gamma(a_3)\Gamma\left(a_1 + a_2 + a_3 - \frac{D}{2}\right)\Gamma\left(\frac{D}{2} - a_1 - a_3\right)\Gamma\left(\frac{D}{2} - a_2 - a_3\right)}{\Gamma\left(D - a_1 - a_2 - a_3\right)},$$

$$\eta_4 = s^{-a_2}\left(-m^2\right)^{\frac{D}{2} - a_1 - a_3} \frac{\Gamma(a_2)\Gamma(a_3)\Gamma\left(a_1 + a_3 - \frac{D}{2}\right)\Gamma\left(\frac{D}{2} - a_2 - a_3\right)}{\Gamma\left(\frac{D}{2} - a_2\right)}.$$

Evaluation III

- Specialize to $a_1 = a_2 = a_3 = 1$ so that

$$H = \int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{x_1 m^2} e^{-\frac{x_1 x_2}{x_1+x_2+x_3} s}}{(x_1+x_2+x_3)^{D/2}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3.$$

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- Then with $D = 4 - 2\epsilon$:

Corollary

For $|\frac{s}{m^2}| < 1$,

$$H = (-m^2)^{-1-\epsilon} \Gamma(\epsilon - 1) {}_2F_1 \left(\begin{matrix} 1+\epsilon, 1 \\ 2-\epsilon \end{matrix} \middle| \frac{s}{m^2} \right).$$

Corollary

For $|\frac{s}{m^2}| > 1$,

$$H = s^{-1-\epsilon} \frac{\Gamma(-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(1 - \frac{m^2}{s}\right)^{-2\epsilon} - m^{-2\epsilon} \frac{\Gamma(\epsilon)}{\epsilon s} {}_2F_1 \left(\begin{matrix} \epsilon, 1 \\ 1-\epsilon \end{matrix} \middle| \frac{m^2}{s} \right).$$

Thoughts

- The method of brackets produces evaluations for the different regions of the kinematic variables.
- Alternative to introducing Mellin-Barnes representations
- Most aspects of this process are automatable.
- Karen Kohl is working on an implementation in SAGE.

Ising Integrals

- Studied by Bailey, Borwein, Crandall:

$$C_{n,k} = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^{k+1}} \frac{\mathrm{d}u_1}{u_1} \cdots \frac{\mathrm{d}u_n}{u_n}$$

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$$= \frac{2^{n-k+1}}{n!\,k!}\int_0^\infty t^k K_0^n(t)\,\mathrm{d}t$$

D. H. Bailey, J. M. Borwein and R. E. Crandall. "Integrals of the Ising class." *Jour. Phys. A*, **39**, 2006.

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$$= \frac{2^{n-k+1}}{n!\,k!} \int_0^\infty t^k K_0^n(t)\,\mathrm{d}t$$

- $C_{1,1} = 2,\ C_{2,1} = 1,\ C_{3,1} = L_{-3}(2),\ C_{4,1} = \frac{7}{12}\zeta(3)$

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D. H. Bailey, J. M. Borwein and R. E. Crandall. "Integrals of the Ising class." *Jour. Phys. A*, **39**, 2006.

Ising Integrals — $n = 2$

$$C_{2,k} = 2 \int_0^\infty \int_0^\infty \frac{\mathrm{d}x \, \mathrm{d}y}{xy \left(x + 1/x + y + 1/y\right)^{k+1}}$$

Ising Integrals — $n = 2$

$$C_{2,k} = 2\int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy\,(x+1/x+y+1/y)^{k+1}}$$
$$= \frac{2}{k!} \sum_{\{n\}} \phi_{\{n\}} \langle n_1+n_2+n_3+n_4+k+1\rangle \langle n_1-n_3\rangle \langle n_2-n_4\rangle$$

Ising Integrals — $n = 2$

$$C_{2,k} = 2\int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy\,(x + 1/x + y + 1/y)^{k+1}}$$
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- 4 indices, 3 brackets: 1 free variable

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- No luck for all choices of free variables.

Ising Integrals — perturbing

- $C_{2,k}$ is the case $\varepsilon \to 0$ of

$$2\int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{x^{1-\varepsilon} y\,(x+1/x+y+1/y)^{k+1}}$$
$$= \frac{2}{k!} \sum_{\{n\}} \phi_{\{n\}} \langle n_1+n_2+n_3+n_4+k+1\rangle \langle n_1-n_3+\varepsilon\rangle \langle n_2-n_4\rangle$$

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- Combined with n_3:

$$\frac{2}{k!}\Gamma(-\varepsilon)\Gamma(1+\tfrac{\varepsilon}{2})^2 {}_2F_1\left(\left.\begin{matrix}1+\frac{\varepsilon}{2}, 1+\frac{\varepsilon}{2}\\ 1+\varepsilon\end{matrix}\right| A\right) + \frac{2}{k!} A^{-\varepsilon}\Gamma(\varepsilon)\Gamma(1-\tfrac{\varepsilon}{2})^2 {}_2F_1\left(\left.\begin{matrix}1-\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\\ 1-\varepsilon\end{matrix}\right| A\right)$$

Ising Integrals — minding form

$$C_{2,k} = 2 \int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy\,(x + 1/x + y + 1/y)^{k+1}}$$

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C_{2,k} &= 2\int_0^\infty \int_0^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy\,(x+1/x+y+1/y)^{k+1}} \\
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&\qquad\qquad \times \langle n_1+n_3+k+1\rangle\,\langle n_1+n_4+k+1\rangle \\
&= \frac{\Gamma(-n_1^*)\,\Gamma(-n_2^*)\,\Gamma(-n_3^*)\,\Gamma(-n_4^*)}{\Gamma(k+1)\Gamma(-n_1^*-n_2^*)}
\end{aligned}$$

- $n_1^* = n_2^* = n_3^* = n_4^* = -\frac{k+1}{2}$

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&\qquad\qquad \times \langle n_1+n_3+k+1\rangle\,\langle n_1+n_4+k+1\rangle \\
&= \frac{\Gamma(-n_1^*)\,\Gamma(-n_2^*)\,\Gamma(-n_3^*)\,\Gamma(-n_4^*)}{\Gamma(k+1)\Gamma(-n_1^*-n_2^*)} = \frac{\Gamma\left(\frac{k+1}{2}\right)^4}{\Gamma(k+1)^2}.
\end{aligned}$$

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Challenges

- The form of the integrand makes a huge difference.
  How can it be automatically optimized?

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- The form of the integrand makes a huge difference.
  How can it be automatically optimized?
- More complicated integrals/bracket series need to be perturbed.
  How to automatize the insertion of the necessary parameters?

The method of brackets

$$\langle s \rangle := \int_0^\infty x^{s-1}\,\mathrm{d}x$$

Rule (Multinomial)

$$\frac{1}{(a_1+a_2+\cdots+a_r)^s} = \sum_{m_1,\ldots,m_r} \phi_{\{m\}} a_1^{m_1}\cdots a_r^{m_r} \frac{\langle s+m_1+\cdots+m_r\rangle}{\Gamma(s)}$$

Rule (Evaluation)

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1,\ldots,n_r)\,\langle a_{11}n_1+\cdots a_{1r}n_r+b_1\rangle\cdots\langle a_{r1}n_1+\cdots a_{rr}n_r+b_r\rangle = \frac{1}{|\det|} f(n_1^*,\ldots,n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*),$$

Rule (Combining)

If there are more summation indices than brackets, free variables are chosen. Each choice produces a series. Those converging in a common region are added.

Ramanujan's Master Theorem

Rule

$$\sum_n \phi_n \lambda(n) \langle an + b \rangle = \frac{1}{|a|} \lambda(n^*) \Gamma(-n^*),$$

where n^* *is the solution of the equation* $an + b = 0$*.*

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- Therefore:

$$\int_0^\infty x^{s-1} f(x)\, \mathrm{d}x = \sum_n \phi_n \lambda(n) \langle n + s \rangle$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda(n) x^n$$

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- This is Ramanujan's Master Theorem.

A joke in the sense of Littlewood

Theorem (Ramanujan's Master Theorem)

$$\int_0^\infty x^{s-1}\left\{\lambda(0)-\frac{x}{1!}\lambda(1)+\frac{x^2}{2!}\lambda(2)-\cdots\right\}\mathrm{d}x=\Gamma(s)\lambda(-s)$$

- Nearly discovered as early as 1847 by Glaisher and O'Kinealy.

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$$= \int_0^\infty x^{s-1} \sum_{n=0}^\infty \frac{(-1)^n}{n!} E^n x^n \, \mathrm{d}x \cdot \lambda(0)$$

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$$=\int_0^\infty x^{s-1}e^{-Ex}\,\mathrm{d}x\cdot\lambda(0)$$

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Rigorous Ramanujan's Master Theorem

Theorem (Ramanujan's Master Theorem)

$$\int_0^\infty x^{s-1} \left\{ \varphi(0) - x\varphi(1) + x^2\varphi(2) - \cdots \right\} \mathrm{d}x = \frac{\pi}{\sin s\pi} \varphi(-s)$$

- Previous form: $\varphi(u) = \lambda(u)/\Gamma(u+1)$

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for $0 < \mathrm{Re}\ s < \delta$*, provided that*

- φ *is analytic (single-valued) on the half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \mathrm{Re}\ u \geq -\delta\},$$

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- φ *is analytic (single-valued) on the half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \operatorname{Re} u \geq -\delta\},$$

- φ *satisfies the growth condition*

$$|\varphi(v + iw)| < Ce^{Pv + A|w|}$$

for some $A < \pi$ *and for all* $v + iw \in H(\delta)$.

- Previous form: $\varphi(u) = \lambda(u)/\Gamma(u+1)$

Multidimensional generalization

Theorem

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1, \ldots, n_r) \langle a_{11} n_1 + \cdots a_{1r} n_r + b_1 \rangle \cdots \langle a_{r1} n_1 + \cdots a_{rr} n_r + b_r \rangle$$
$$= \frac{1}{|\det(A)|} f(n_1^*, \ldots, n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*),$$

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Converting brackets back:

$$\int_0^\infty \int_0^\infty \sum_{n_1, n_2} \phi_{n_1, n_2} f(n_1, n_2) x^{a_{11}n_1 + a_{12}n_2 + b_1 - 1} y^{a_{21}n_1 + a_{22}n_2 + b_2 - 1} \, \mathrm{d}x \, \mathrm{d}y$$

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Substitute $(u,v) = (x^{a_{11}}y^{a_{21}}, x^{a_{12}}y^{a_{22}})$ with $\frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\frac{\mathrm{d}u\,\mathrm{d}v}{uv}$:

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$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) u^{n_1-n_1^*-1} v^{n_2-n_2^*-1}\,\mathrm{d}u\,\mathrm{d}v$$

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Substitute $(u,v) = (x^{a_{11}}y^{a_{21}}, x^{a_{12}}y^{a_{22}})$ with $\frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}\frac{\mathrm{d}u\,\mathrm{d}v}{uv}$:

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|} \int_0^\infty \int_0^\infty \sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2) u^{n_1-n_1^*-1} v^{n_2-n_2^*-1}\,\mathrm{d}u\,\mathrm{d}v$$

$$= \frac{1}{|\det|} f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*)$$

The Mellin transform

- The Mellin transform of $f(x)$ is $F(s) = \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$.
- Mellin inversion: $f(x) = \dfrac{1}{2\pi i} \displaystyle\int_{c-i\infty}^{c+\infty} F(s) x^{-s}\,\mathrm{d}s$
- Parseval's identity: $\displaystyle\int_0^\infty f(x) g(x)\,\mathrm{d}x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(1-s)\,\mathrm{d}s$

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Example

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\lambda(-s)x^{-s}\,\mathrm{d}s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\lambda(n)x^n$$

and so

$$\int_0^\infty x^{s-1} f(x)\,\mathrm{d}x = \lambda(-s)\Gamma(s).$$

The method of brackets the Mellin way

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$$\frac{\Gamma(s)}{(x_1+x_2)^s} = \sum_{m_1,m_2} \phi_{m_1,m_2} x_1^{m_1} x_2^{m_2} \langle s + m_1 + m_2 \rangle$$

has its counterpart as

$$\frac{\Gamma(s)}{(x_1+x_2)^s} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x_1^z x_2^{-z-s} \Gamma(-z) \Gamma(z+s) \, \mathrm{d}z.$$

Distributional Mellin transform

- Test functions $\phi \in \mathcal{T}(a,b)$ with $\phi \in C^\infty(0,\infty)$ and

$$x^k \phi^{(k)}(x) \underset{x \to 0}{=} o\left(x^{a+\varepsilon_a-1}\right), \qquad x^k \phi^{(k)}(x) \underset{x \to \infty}{=} o\left(x^{b-\varepsilon_b-1}\right).$$

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- $\mathcal{M}[f;s]$ is a holomorphic function for $a < \operatorname{Re} s < b$.

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$$\langle f, g \rangle = \int_0^\infty f(x) g(x)\,\mathrm{d}x = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \tilde{G}(s)\,\mathrm{d}s =: \left\langle F, \tilde{G} \right\rangle_M$$

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- Thus $\langle s\rangle = \mathcal{M}\left[1; s\right] = \delta(s)$ is the Dirac distribution.

Distributional bracket in action

Theorem

Under appropriate conditions on λ, we have the rule

$$\sum_n \phi_n \lambda(n) \langle n+s \rangle = \Gamma(s)\lambda(-s).$$

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$$= \sum_n \phi_n \lambda(n) \chi(-n).$$

The end

THANK YOU!

The method of brackets

$$\langle s \rangle := \int_0^\infty x^{s-1}\,\mathrm{d}x$$

Rule (Multinomial)

$$\frac{1}{(a_1 + a_2 + \cdots + a_r)^s} = \sum_{m_1,\ldots,m_r} \phi_{\{m\}} a_1^{m_1} \cdots a_r^{m_r} \frac{\langle s + m_1 + \cdots + m_r \rangle}{\Gamma(s)}$$

Rule (Evaluation)

$$\sum_{\{n\}} \phi_{\{n\}} f(n_1,\ldots,n_r) \langle a_{11}n_1 + \cdots a_{1r}n_r + b_1 \rangle \cdots \langle a_{r1}n_1 + \cdots a_{rr}n_r + b_r \rangle = \frac{1}{|\det|} f(n_1^*,\ldots,n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*),$$

Rule (Combining)

If there are more summation indices than brackets, free variables are chosen. Each choice produces a series. Those converging in a common region are added.