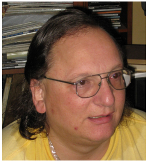
How far does a drunkard get?

Graduate Student Colloquium

Armin Straub

Tulane University, New Orleans

April 12, 2011




Joint with: Jon Borwein (U. of Newcastle, AU), Dirk Nuyens (K.U.Leuven, BE), James Wan (U. of Newcastle, AU), Wadim Zudilin (U. of Newcastle, AU)

Random walks in the plane

- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

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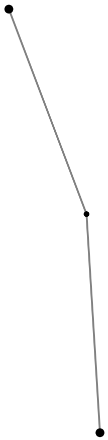

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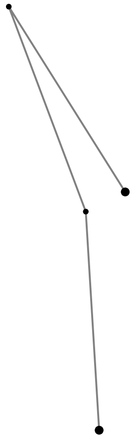

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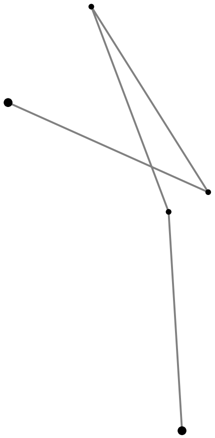

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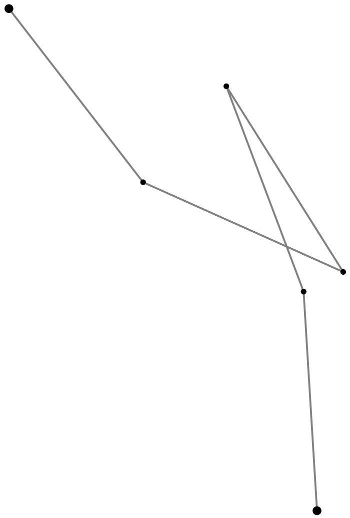
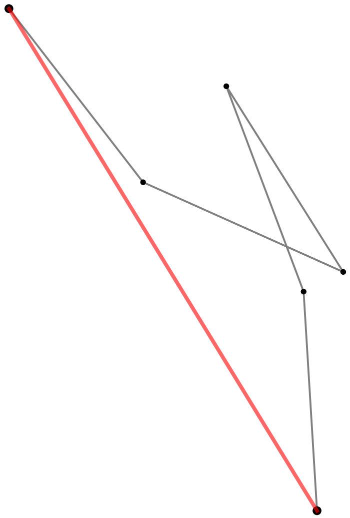


Random walks in the plane

- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.
- We are interested in the distance traveled in n steps.

  For instance, how large is this distance on average?



How the random walk got its name

- Asked by Karl Pearson in Nature in 1905

The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

KARL PEARSON.

The Gables, East Ilsley, Berks.

K. Pearson. "The random walk." *Nature*, **72**, 1905.

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The Problem of the Random Walk.

THIS problem, proposed by Prof. Karl Pearson in the current number of NATURE, is the same as that of the composition of n iso-periodic vibrations of unit amplitude and of phases distributed at random, considered in *Phil. Mag.*, x., p. 73, 1880; xlvii., p. 246, 1899; ("Scientific Papers," i., p. 491, iv., p. 370). If n be very great, the probability sought is

$$\frac{2}{n} e^{-r^2/n} r\,dr.$$

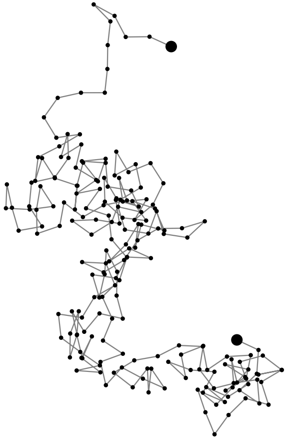
Probably methods similar to those employed in the papers referred to would avail for the development of an approximate expression applicable when n is only moderately great.

RAYLEIGH.

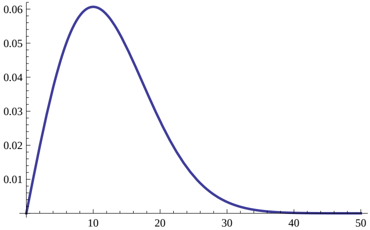
Terling Place, July 29.

K. Pearson. "The ran

Long walks



- For long walks, the probability density is approximately $\frac{2x}{n}e^{-x^2/n}$
- For instance, for $n = 200$:



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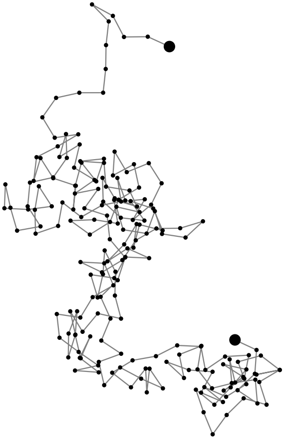

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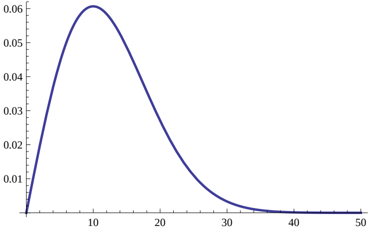
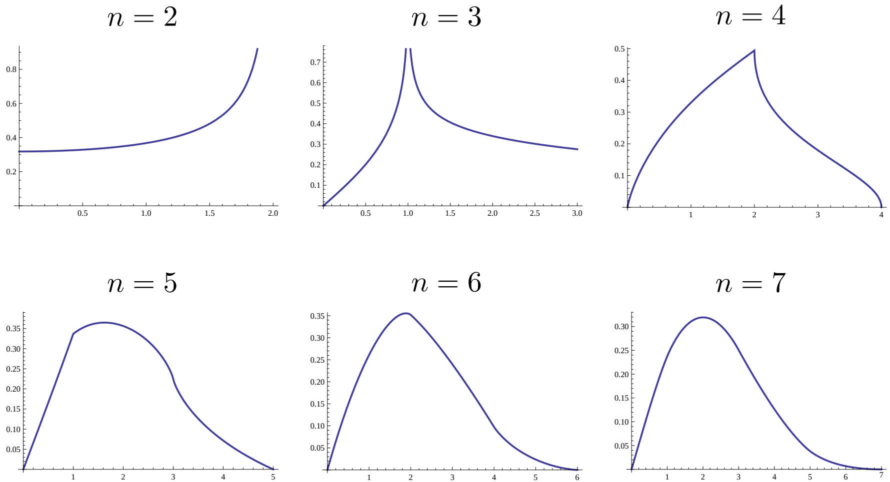


The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point! KARL PEARSON.

Densities


$n = 2$
0.2 0.4 0.6 0.8
0.5 1.0 1.5 2.0
$n = 3$
0.1 0.2 0.3 0.4 0.5 0.6 0.7
0.5 1.0 1.5 2.0 2.5 3.0
$n = 4$
0.1 0.2 0.3 0.4 0.5
1 2 3 4
$n = 5$
0.05 0.10 0.15 0.20 0.25 0.30 0.35
1 2 3 4 5
$n = 6$
0.05 0.10 0.15 0.20 0.25 0.30 0.35
1 2 3 4 5 6
$n = 7$
0.05 0.10 0.15 0.20 0.25 0.30
1 2 3 4 5 6 7

Densities

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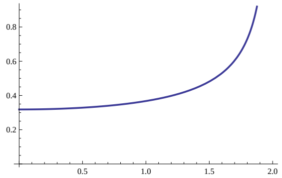
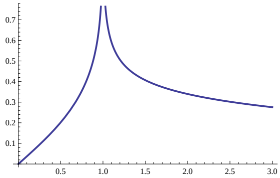
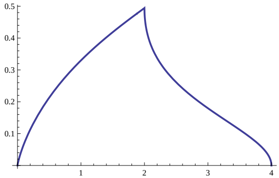


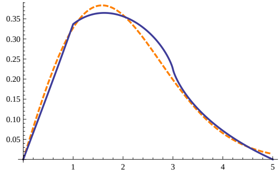
$n = 3$



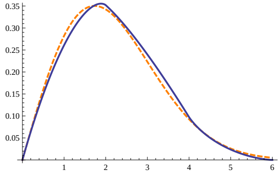
$n = 4$



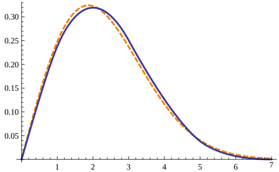
$n = 5$



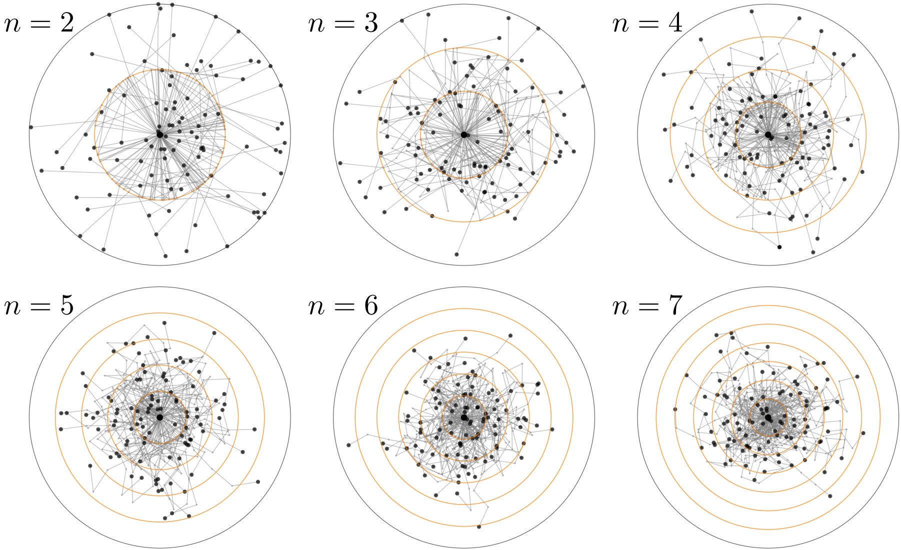
$n = 6$



$n = 7$

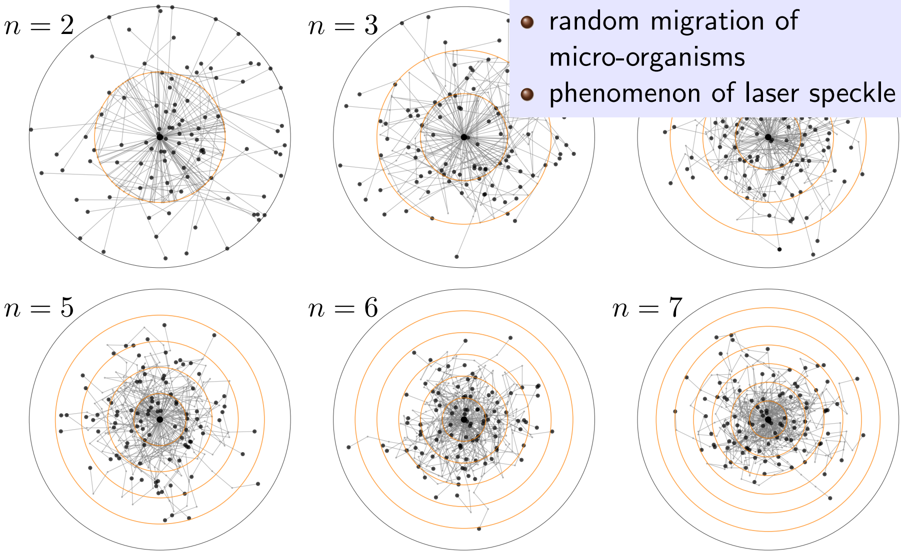


Hornets gone wild



Hornets gone wild

- dispersion of mosquitoes
- random migration of micro-organisms
- phenomenon of laser speckle



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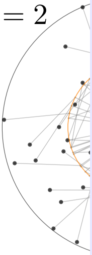

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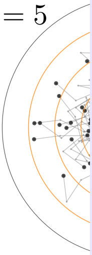

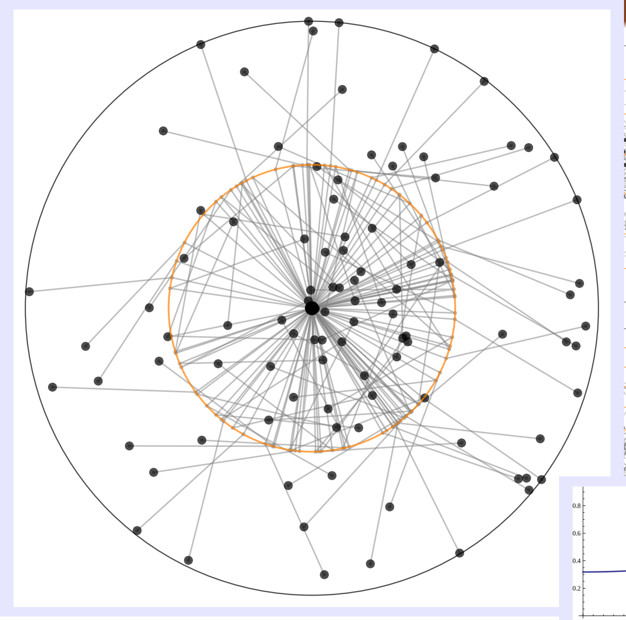

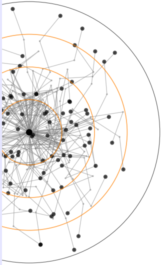

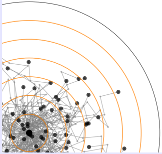

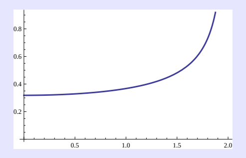

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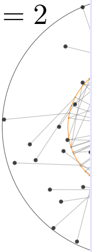

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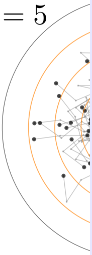

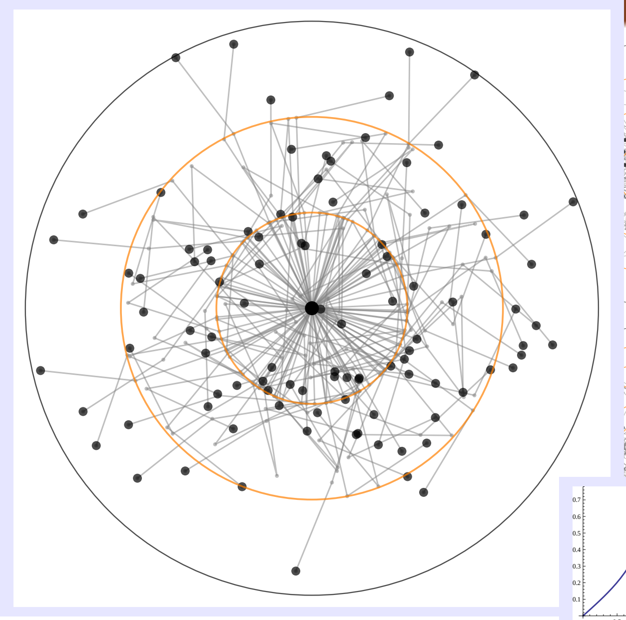

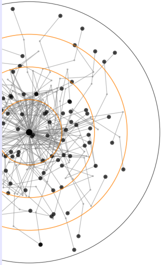

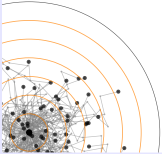

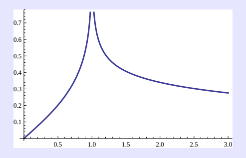

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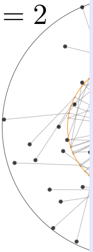

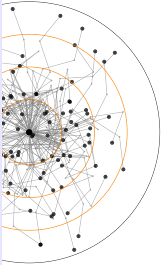

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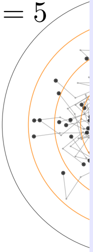

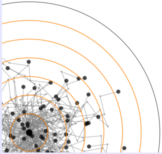

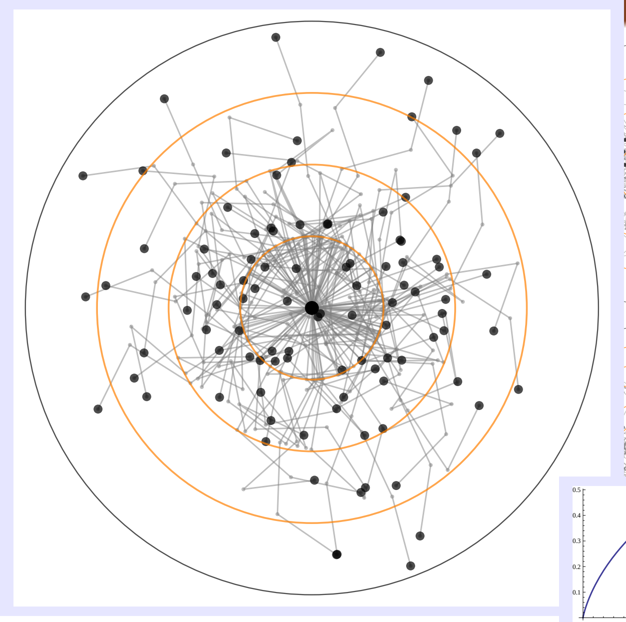

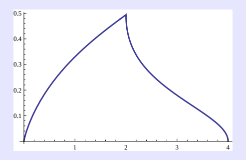

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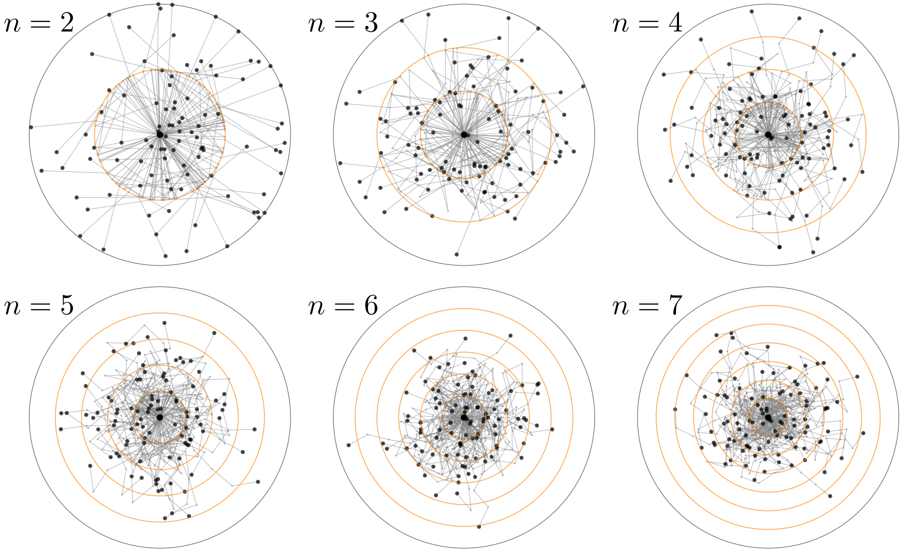

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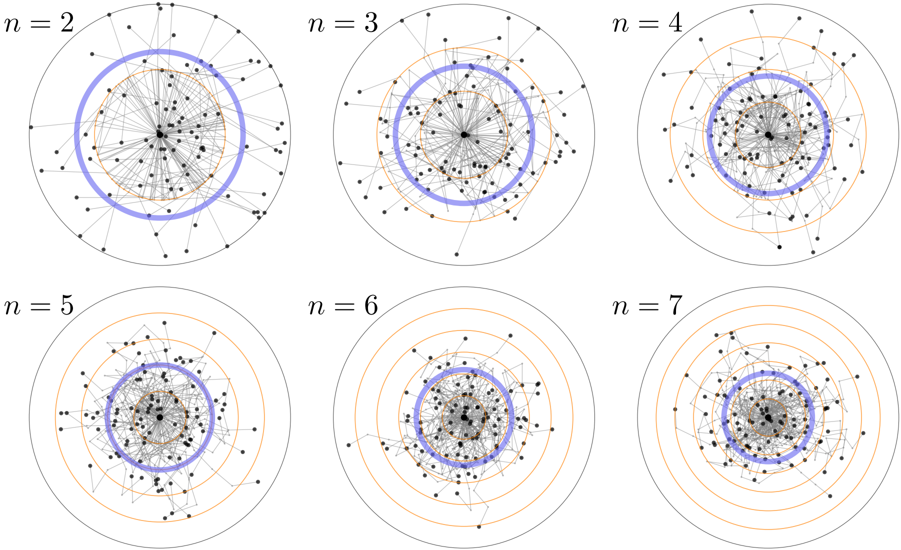
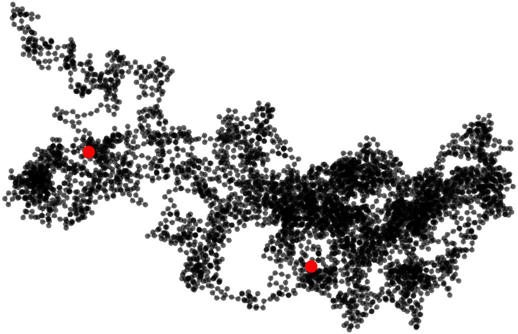


Drunken birds



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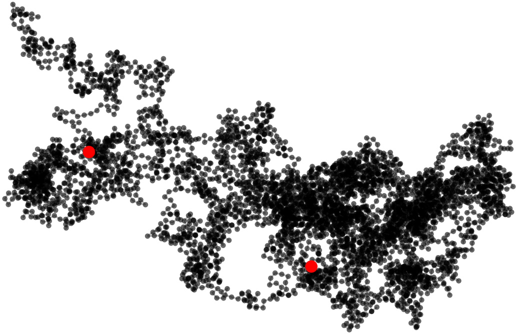

A drunk man will find his way home,
but a drunk bird may get lost forever.

— Shizuo Kakutani

Moments

- The moments of a RV X are $E(X)$, $E(X^2)$, $E(X^3)$, ...
- If X has probability density $f(x)$ then

$$E(X^s) = \int_{-\infty}^{\infty} x^s f(x)\,\mathrm{d}x$$

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Fact

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Assumption: for instance, $f(x)$ compactly supported

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- $\int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$ is called the Mellin transform of f

Moments of the random walks

- Represent the kth step by the complex number $e^{2\pi i x_k}$.
- The distance after n steps is $\left|\sum_{k=1}^{n} e^{2\pi i x_k}\right|$.

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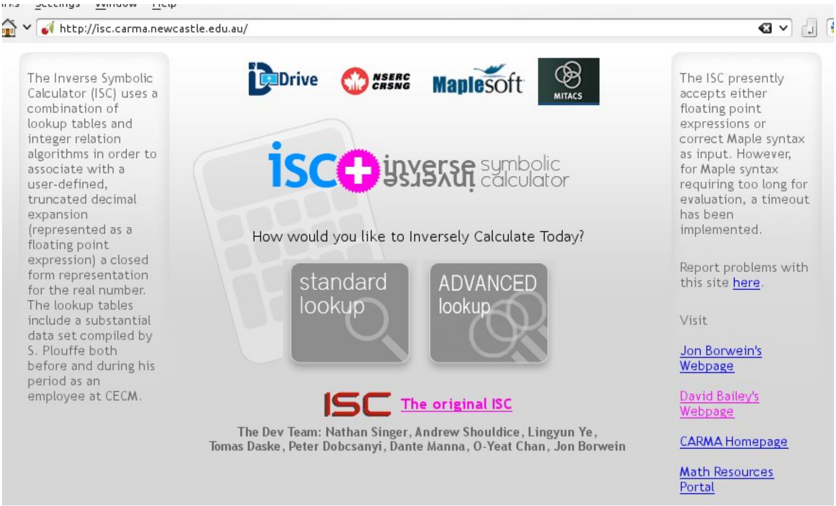
- The distance after n steps is $\left|\sum_{k=1}^n e^{2\pi i x_k}\right|$.
- The sth moment of the distance after n steps is:

$$W_n(s) := \int_{[0,1]^n} \left|\sum_{k=1}^n e^{2\pi x_k i}\right|^s \mathrm{d}\boldsymbol{x}$$

  In particular, $W_n(1)$ is the average distance after n steps.
- Trivially $W_1(s) = 1$.

Average distance traveled in two steps

- Numerically: $W_2(1) \approx 1.27323954473516268 62$



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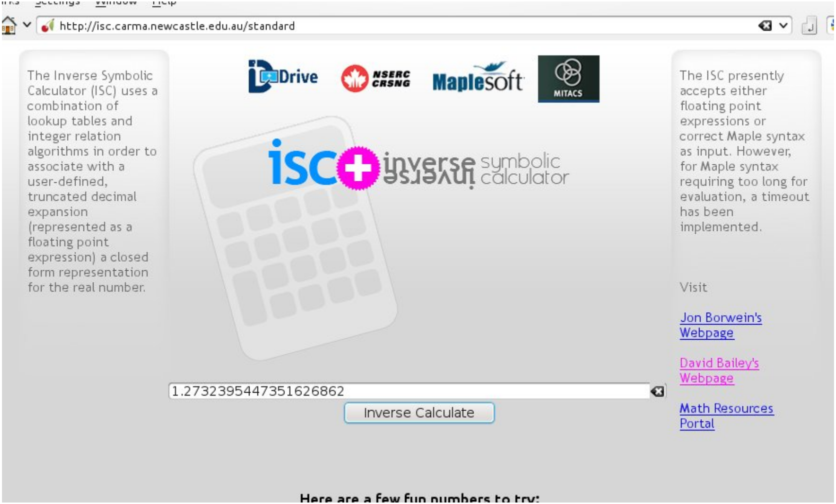

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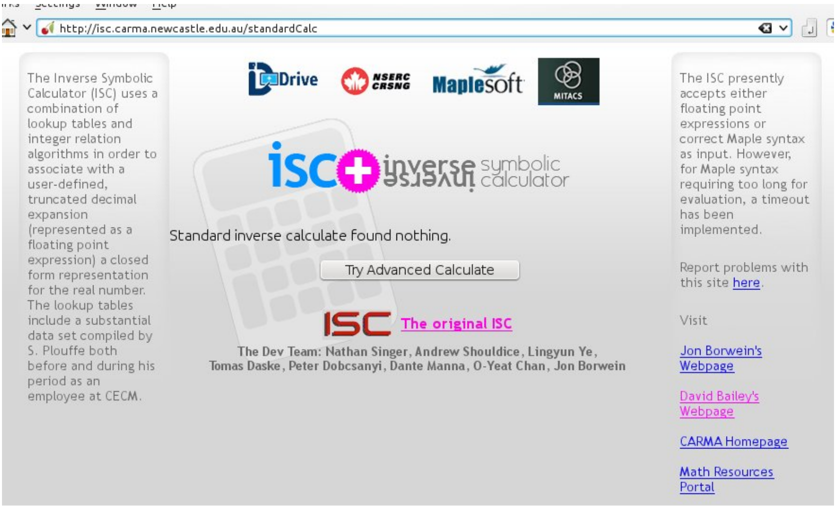

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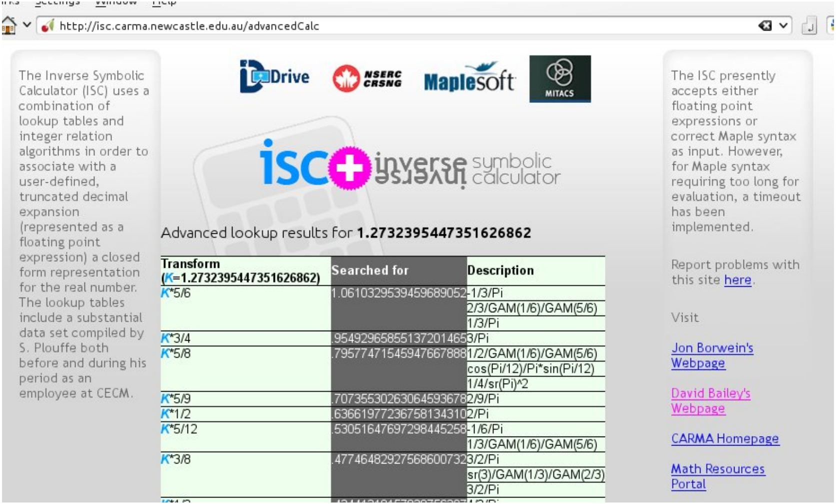

http://isc.carma.newcastle.edu.au/advancedCalc

The Inverse Symbolic Calculator (ISC) uses a combination of lookup tables and integer relation algorithms in order to associate with a user-defined, truncated decimal expansion (represented as a floating point expression) a closed form representation for the real number. The lookup tables include a substantial data set compiled by S. Plouffe both before and during his period as an employee at CECM.

isc inverse symbolic calculator

Advanced lookup results for **1.2732395447351626862**

Transform (K=1.2732395447351626862)	Searched for	Description
K*5/6	1.0610329539459689052	-1/3/Pi
		2/3/GAM(1/6)/GAM(5/6)
		1/3/Pi
K*3/4	.95492965855137201465	3/Pi
K*5/8	.79577471545947667888	1/2/GAM(1/6)/GAM(5/6)
		cos(Pi/12)/Pi*sin(Pi/12)
		1/4/sr(Pi)^2
K*5/9	.70735530263064593678	2/9/Pi
K*1/2	.63661977236758134310	2/Pi
K*5/12	.53051647697298445258	-1/6/Pi
		1/3/GAM(1/6)/GAM(5/6)
K*3/8	.47746482927568600732	3/2/Pi
		sr(3)/GAM(1/3)/GAM(2/3)
		3/2/Pi

The ISC presently accepts either floating point expressions or correct Maple syntax as input. However, for Maple syntax requiring too long for evaluation, a timeout has been implemented.

Report problems with this site here.

Visit

Jon Borwein's Webpage

David Bailey's Webpage

CARMA Homepage

Math Resources Portal

The simple two-step case confirmed

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

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- There is always a 1-dimensional reduction:

$$W_2(1) = \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y = \frac{4}{\pi} \approx 1.27324$$

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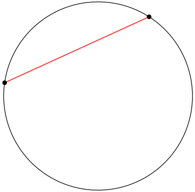
- Mathematica 7 and Maple 14 think the answer is 0.
- There is always a 1-dimensional reduction:

$$W_2(1) = \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y = \frac{4}{\pi} \approx 1.27324$$

- This is the average length of a random arc on a unit circle.



The average distance for 3 and more steps

- $W_n(s) := \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$
- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

- In fact, $W_5(1) \approx 2.0081618$ was the best estimate we could compute directly, notwithstanding the availability of 256 cores at the Lawrence Berkeley National Laboratory.

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- Hard to evaluate numerically to high precision. For instance, Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.
- Closed forms as in the case $n = 2$?

Can we guess $W_3(1)$?

- $W_3(1) = 1.57459723755189365749\ldots$

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Idea

If we suspect that a number x_0 can be written as $x_0 = a_1 x_1 + \ldots a_n x_n$ for other numbers x_j and rational a_j then this can be detected!

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- PSLQ takes numbers $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and tries to find integers $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, not all zero, such that

$$\mathbf{x} \cdot \mathbf{m} = m_1 x_1 + \ldots + m_n x_n = 0.$$

The vector $\mathbf{m}$ is called an integer relation for $\mathbf{x}$. In case that no relation is found, PSLQ provides a lower bound for the norm of any potential integer relation.

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```
In[1]:= << "~/docs/math/mathematica/pslq.m"

    Basic PSLQ implementation by Armin Straub
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        -- Tulane University -- Version 1.2 (2010/12/17)

In[2]:= W2 = 1.2732395447351626861510701069801148962756771659236515899813387524711743810738122807209;
        W3 = 1.5745972375518936574946921830765196902216661807585191701936930983018311805944543821311;

In[4]:= PSLQ[{W2, 1, 1 / Pi, 1 / Pi ^ 2}]

Out[4]= {1, 0, -4, 0}
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        PSLQ::lowprec : Precision too low to continue (155 iterations performed).

        PSLQ::norel : No integer relation was found. The norm of any true integer relation is at least 1.3248876487095543`*^13.

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```

Getting data: computing some moments

The sth moment of the distance after n steps:

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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Even moments

n	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	2	6	20	70	252	A000984
3	3	15	93	639	4653	A002893
4	4	28	256	2716	31504	A002895
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- Apparently: $W_n(2) = n$
- Also: $W_n(10) \equiv n$ modulo 10

The integer sequence database

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Integer Sequences

1,4,28,256 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,4,28,256**

Displaying 1-2 of 2 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A064340 Generalized Catalan numbers C(2,2; n). +20 10

1, **1**, **4**, **28**, **256**, 2704, 31168, 380608, 4840960, 63458560, 851399680, 11635096576, 161396604928, 2266669453312, 32166082822144, 460531091685376, 6644185553305600, 96498260064403456, 1409750653282287616 (list; graph; listen; history; internal format)

OFFSET 0,3

COMMENTS See triangle A064879 with columns m built from C(m,m; n), m >= 0, also for Derrida et al. and Liggett references.

FORMULA a(n)= ((4^(n-1))/(n-1))*sum((m+1)*(m+2)*binomial(2*(n-2)-m, n-2-m)*((1/2)^(m+1)), m=0..n-2), n >= 2, a(0) := 1=: a(1).
G.f.:(1-3*x*c(4*x))/(1-2*x*c(4*x))^2 = c(4*x)*(3+c(4*x))/(1+c(4*x))^2 = (1+5*x+3*c(4*x)*(2*x)^2)/(1+2*x)^2 with c(x)= A(x) g.f. of Catalan numbers A000108.

CROSSREFS A000108 (Catalan as C(1, 1, n)).

KEYWORD nonn,easy

AUTHOR Wolfdieter Lang (wolfdieter.lang(AT)physik.uni-karlsruhe.de), Oct 12 2001

A002895 Number of 2n-step polygons on diamond lattice. (Formerly M3626 N1473) +20 4

1, **4**, **28**, **256**, 2716, 31504, 387136, 4951552, 65218204, 878536624,

The integer sequence database

A002895 Number of 2n-step polygons on diamond lattice. (Formerly M3626 N1473) +20 4

1, **4**, **28**, **256**, 2716, 31504, 387136, 4951552, 65218204, 878536624, 12046924528, 167595457792, 2359613230144, 33557651538688, 481365424895488, 6956365106016256, 101181938814289564, 1480129751586116848 (list; graph; listen; history; internal format)

OFFSET 0,2

COMMENTS a(n) is the (2n)th moment of the distance from the origin of a 4-step random walk in the plane - Peter M.W. Gill (peter.gill(AT)nott.ac.uk), Mar 03 2004

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David H. Bailey, Jonathan M. Borwein, David Broadhurst and M. L. Glasser, Elliptic integral evaluations of Bessel moments, arXiv:0801.0891.
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J. A. Hendrickson, Jr., On the enumeration of rectangular (0,1)-matrices, Journal of Statistical Computation and Simulation, 51 (1995), 291-313.
N. J. A. Sloane, A Handbook of Integer Sequences, Academic Press, 1973 (includes this sequence).
N. J. A. Sloane and Simon Plouffe, The Encyclopedia of Integer Sequences, Academic Press, 1995 (includes this sequence).

LINKS
Jonathan M. Borwein, Dirk Nuyens, Armin Straub and James Wan, Random Walk Integrals, 2010.
L. B. Richmond, J. Shallit, Counting Abelian Squares, arXiv:0807.5028 [Math.CO]. [From R. J. Mathar (mathar(AT)strw.leidenuniv.nl), Oct 30 2008]

FORMULA
Sum_{k=0..n} binomial(n, k)^2 binomial(2n-2k, n-k) binomial(2k, k).
n^3*a(n) = 2*(2*n-1)*(5*n^2-5*n+2)*a(n-1)-64*(n-1)^3*a(n-2). - Vladeta Jovovic (vladeta(AT)eunet.rs), Jul 16 2004
Sum_{n>=0} a(n)*x^n/n!^2 = BesselI(0, 2*sqrt(x))^4. -

The integer sequence database

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Integer Sequences

1,5,45,545,7885 | Search | Hints

(Greetings fr[Search Query]e Encyclopedia of Integer Sequences!)

Search: **seq:1,5,45,545,7885**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created   Format: long | short | data

A169714   The function W_5(2n) (see Borwein et al. reference for definition).   +20 1

1, 5, 45, 545, 7885, 127905 (list; graph; listen; history; internal format)

OFFSET	0,2
LINKS	Jonathan M. Borwein, Dirk Nuyens, Armin Straub and James Wan, Random Walk Integrals, 2010.
KEYWORD	nonn
AUTHOR	N. J. A. Sloane (njas(AT)research.att.com), Apr 17 2010

page 1

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Last modified March 27 11:55 EDT 2011. Contains 186889 sequences.

A combinatorial formula for the even moments

Theorem (Borwein-Nuyens-S-Wan)

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2.$$

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- $f_n(k) := W_n(2k)$ counts the number of *abelian squares*: strings xy of length $2k$ from an alphabet with n letters such that y is a permutation of x.
- Introduced by Erdős and studied by others.
- Surely: $f_1(k) = 1$.

Example

$acbc\ ccba$ is an abelian square. It contributes to $f_3(4)$.

A miracle?

Example

In the case of $n = 2$ we count abelian squares made from two letters.

$$b\,a\,b\,a\,a \quad a\,b\,a\,a\,b.$$

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It follows that $f_2(k) = \binom{2k}{k}$.

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Recall:

$$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$$

$$\Gamma(s+1) = s\Gamma(s)$$

$$\Gamma(1/2) = \sqrt{\pi}$$

- So: $W_2(2k) = \binom{2k}{k}$

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- Putting $k = \frac{1}{2}$ we obtain $\binom{1}{1/2} = \frac{1!}{(1/2)!^2} = \frac{1}{\Gamma^2(3/2)} = \frac{4}{\pi}$
- Indeed: $W_2(s) = \binom{s}{s/2}$

Other combinatorial consequences

- Convolutions:

$$f_{n+m}(k) = \sum_{j=0}^{k} \binom{k}{j}^2 f_n(j)\, f_m(k-j).$$

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- Recursions by Sister Celine, e.g.:

$$(k+2)^2 f_3(k+2) - (10k^2+30k+23) f_3(k+1) + 9(k+1)^2 f_3(k) = 0.$$

Functional equations

- For integers $k \geqslant 0$,

$$(k+2)^2 W_3(2k+4) - (10k^2+30k+23) W_3(2k+2) + 9(k+1)^2 W_3(2k) = 0.$$

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Theorem (Carlson)

If $f(z)$ is analytic for $\mathrm{Re}\,(z) \geqslant 0$, "nice", and

$$f(0)=0, \quad f(1)=0, \quad f(2)=0, \quad \ldots,$$

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$|f(z)| \leqslant A e^{\alpha |z|}$, and

$|f(iy)| \leqslant B e^{\beta |y|}$ for $\beta < \pi$

Functional Equations for $W_n(s)$

- So we get complex functional equations like

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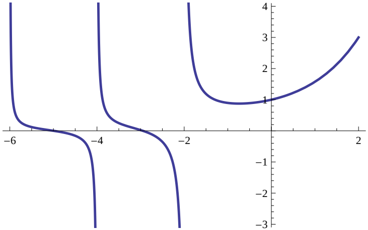
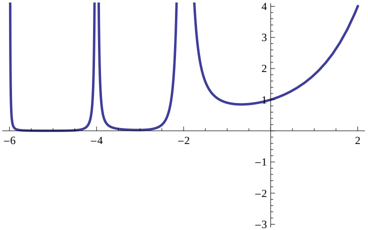
- This gives analytic continuations of $W_n(s)$ to the complex plane, with poles at certain negative integers.

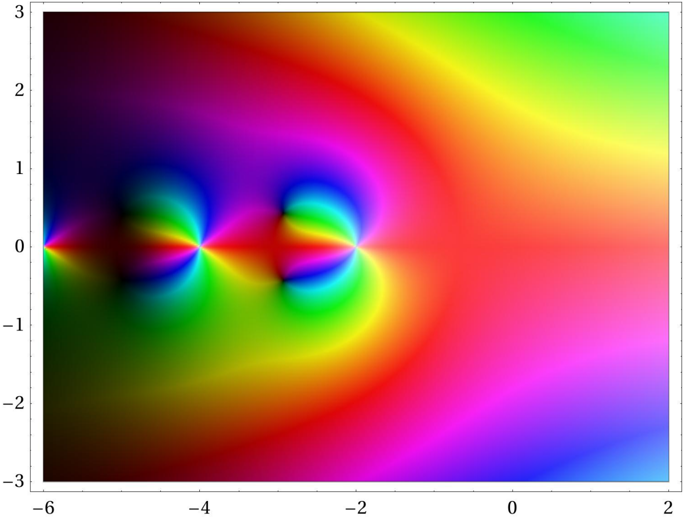


$W_3(s)$

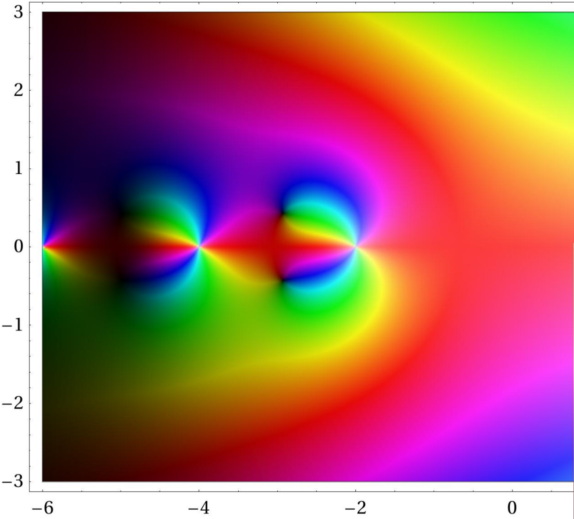


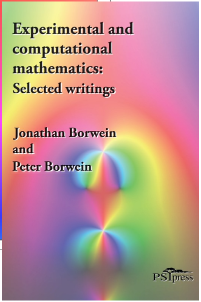
$W_4(s)$

$W_4(s)$ in the complex plane



$W_4(s)$ in the complex plane





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- Idea: again, replace k by a complex variable

Interlude: hypergeometric functions

- The hypergeometric function:

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} $$

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$$ r(n) = \frac{(n+a_1)\cdots(n+a_p)}{(n+b_1)\cdots(n+b_q)} \frac{x}{n+1} $$

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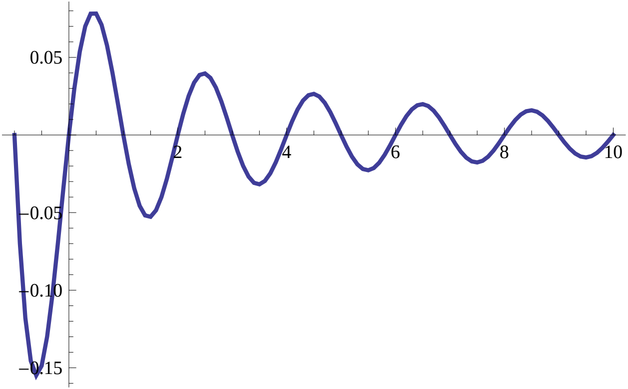
- So by Carlson's Theorem $W_3(s) = V_3(s)$, no!?!??

$W_3(1) = 1.57459723755189\ldots = ?$

- Here's $\mathrm{Re}\,(W_3(s) - V_3(s))$:



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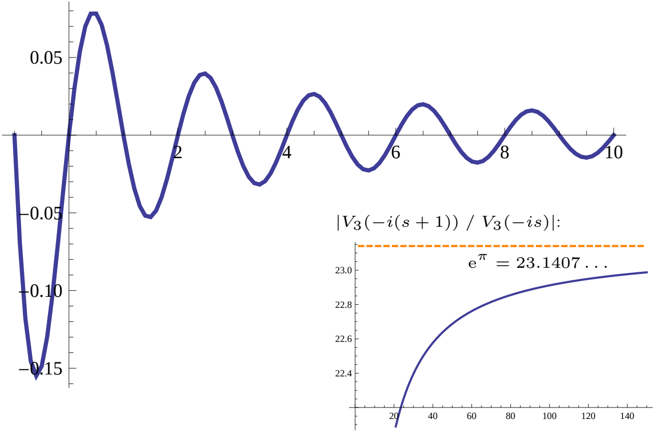

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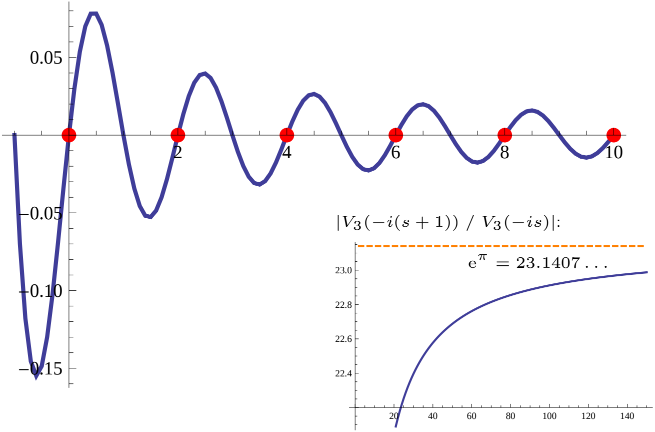

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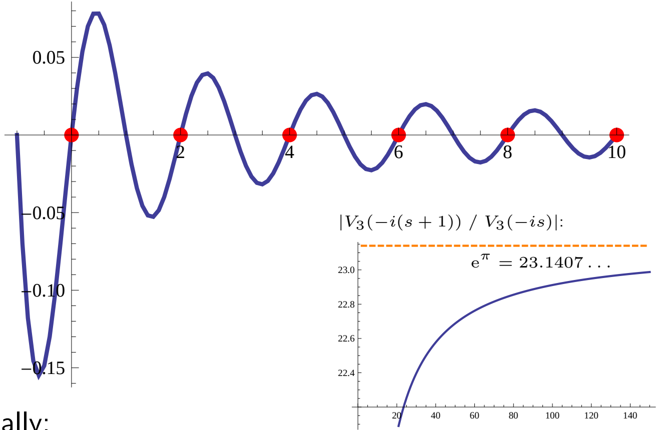

- Numerically:
  $V_3(1) \approx 1.574597 - .126027i$

$W_3(1) = 1.57459723755189\ldots = ?$

Theorem (Borwein-Nuyens-S-Wan)

For integers k we have $W_3(k) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -\frac{k}{2}, -\frac{k}{2} \\ 1, 1\end{matrix}\right| 4\right)$.

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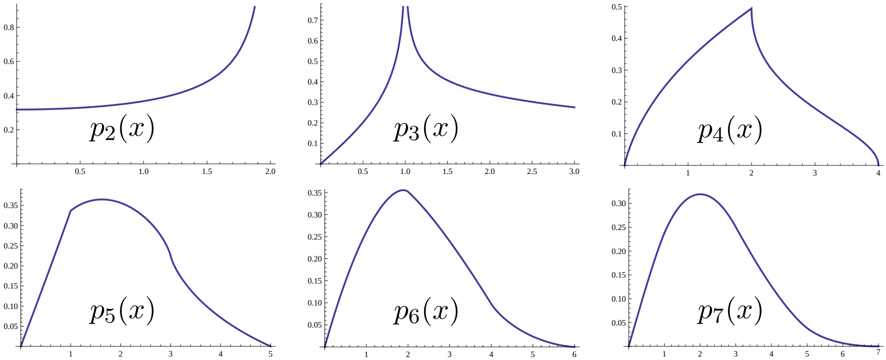
Corollary (Borwein-Nuyens-S-Wan)

$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

- Similar formulas for $W_3(3), W_3(5), \ldots$

Densities


0.8
0.6
0.4
0.2
0.5 1.0 1.5 2.0
$p_2(x)$
0.7
0.6
0.5
0.4
0.3
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0.1
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0.5
0.4
0.3
0.2
0.1
1 2 3 4
$p_4(x)$
0.35
0.30
0.25
0.20
0.15
0.10
0.05
1 2 3 4 5
$p_5(x)$
0.35
0.30
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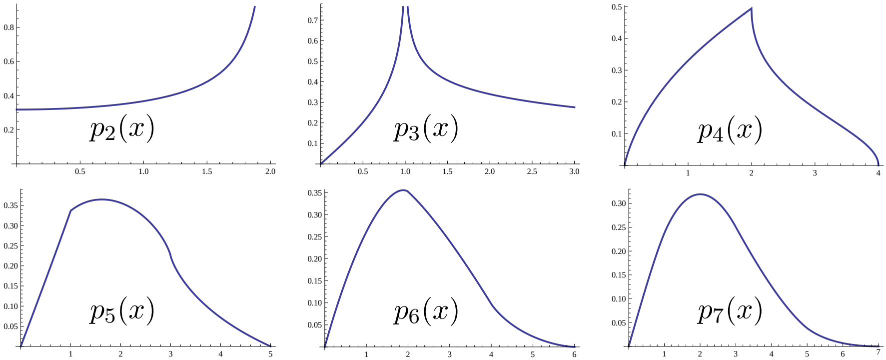
0.20
0.15
0.10
0.05
1 2 3 4 5 6
$p_6(x)$
0.30
0.25
0.20
0.15
0.10
0.05
1 2 3 4 5 6 7
$p_7(x)$

Densities



- p_4 and p_5 are C^0
- p_6 and p_7 are C^1
- p_{2n+4}, p_{2n+5} are C^n

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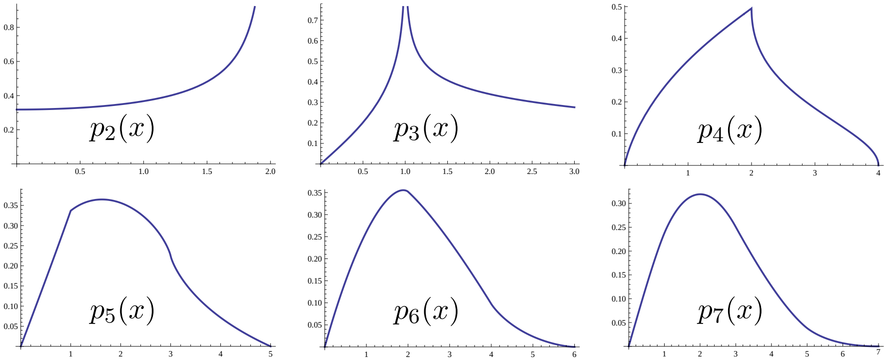

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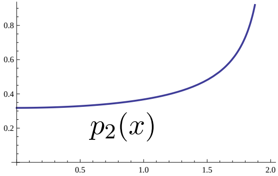

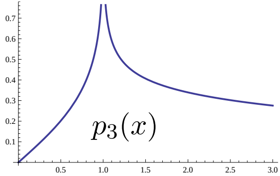

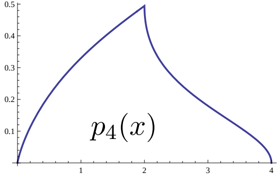

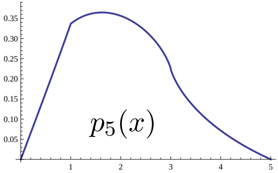

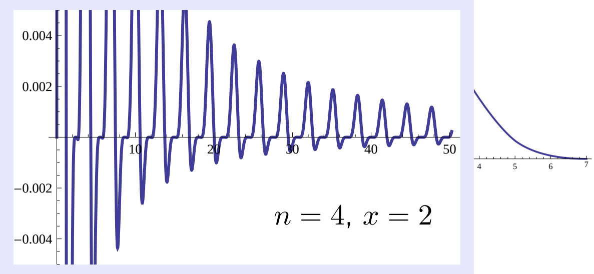

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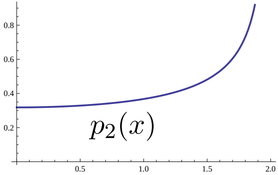
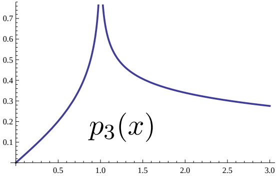
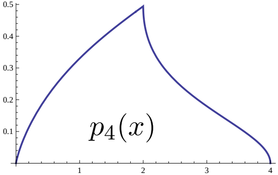
Hypergeometric formulae



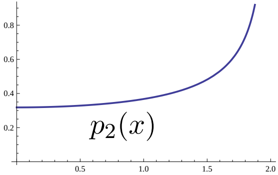


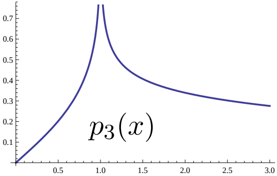


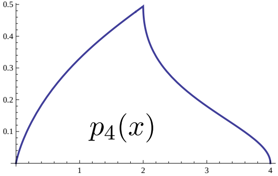
$$p_2(x) = \frac{2}{\pi\sqrt{4 - x^2}}$$

easy

Hypergeometric formulae







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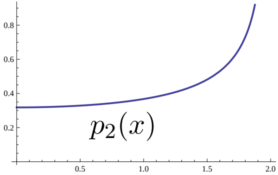

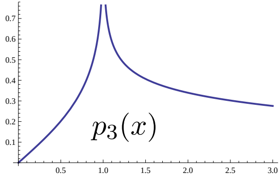

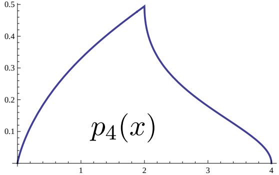

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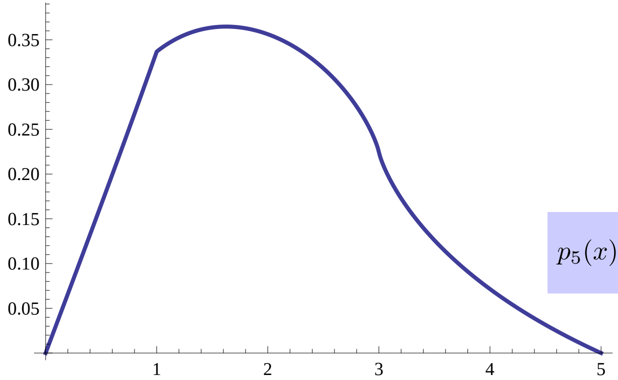
A straight line?



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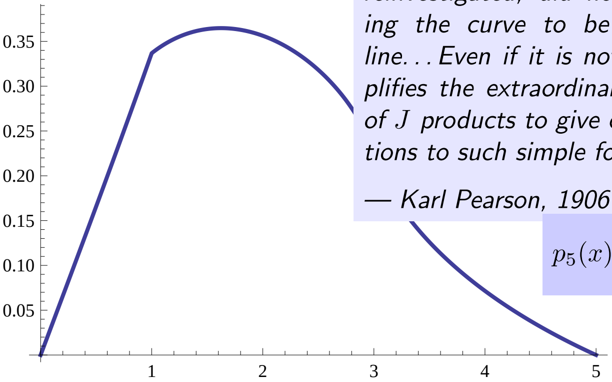

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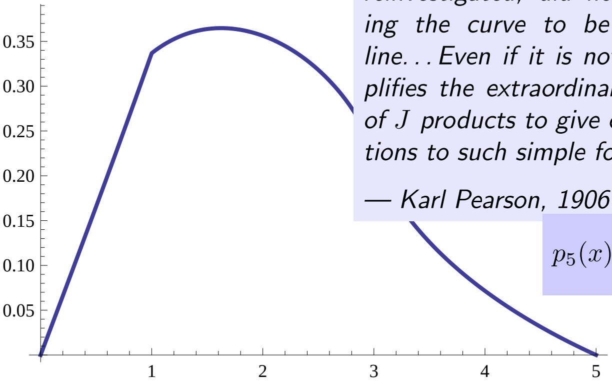

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$$p_5(x) = 0.32993x + 0.0066167x^3 + 0.00026233x^5 + 0.000014119x^7 + O(x^9)$$

Relation between densities and moments

- $W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$
- Or: $W_n(s-1) = \mathcal{M}[p_n; s]$

Mellin transform $F(s)$ of $f(x)$:

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Example

$$(s+4)^3 W_4(s+4) - 4(s+3)(5s^2+30s+48)W_4(s+2) + 64(s+2)^3 W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ where A_4 is

$$(x-4)(x-2)x^3(x+2)(x+4)D_x^3 + 6x^4\left(x^2-10\right)D_x^2 + \\ +x\left(7x^4 - 32x^2 + 64\right)D_x + \left(x^2-8\right)\left(x^2+8\right)$$

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Example

… $W_4(s) = 0$

… $+$

… $3)$

Pole structure of $W_n(s)$ determines $p_n(x)$ at $x = 0$:

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

implies

$$p_4(x) = -\frac{3}{2\pi^2} x \log(x) + \frac{9\log 2}{2\pi^2} x + O(x^3) \quad \text{as } x \to 0$$

Densities in general

Theorem

- *The density p_n satisfies a DE of order $n-1$.*
- *Let $n \leqslant 1000$. If n is even (odd) then p_n is real analytic except at 0 and the even (odd) integers $m \leq n$.*

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Conjecture (confirmed, e.g., for $n \leqslant 1000$)

$$\sum_{\substack{0\leqslant m_1,\ldots,m_j<n/2\\ m_i<m_{i+1}}} \prod_{i=1}^{j} (n-2m_i)^2 = \sum_{\substack{1\leqslant \alpha_1,\ldots,\alpha_j\leqslant n\\ \alpha_i\leqslant \alpha_{i+1}-2}} \prod_{i=1}^{j} \alpha_i(n+1-\alpha_i).$$

Example

$$\sum_{m=0}^{n/2-1} (n-2m)^2 = \sum_{\alpha=1}^{n} \alpha(n+1-\alpha) = \binom{n+2}{3}$$

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$$\sum_{m_1=0}^{n/2-1} \sum_{m_2=0}^{m_1-1} (n-2m_1)^2(n-2m_2)^2$$

$$= \sum_{\alpha_1=1}^{n} \sum_{\alpha_2=1}^{\alpha_1-2} \alpha_1(n+1-\alpha_1)\alpha_2(n+1-\alpha_2)$$

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Outlook: Mahler measure

- Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

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$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

- $W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$

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- $W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$
- Rediscovered the classical results:

$$\mu(1 + x_1 + x_2) = \frac{1}{2} \operatorname{Ls}_2\left(\frac{\pi}{3}\right)$$

$$\mu(1 + x_1 + x_2 + x_3) = \frac{7\zeta(3)}{2\pi^2}$$

Outlook: Log-sine integrals

- Generalized log-sine integral:

$$\mathrm{Ls}_n^{(k)}(\sigma) := -\int_0^\sigma \theta^k \log^{n-1-k} \left|2 \sin \frac{\theta}{2}\right| \mathrm{d}\theta$$

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$$-\mathrm{Ls}_6^{(1)}(\pi) = 24\,\mathrm{Li}_{3,1,1,1}(-1) - 18\,\mathrm{Li}_{5,1}(-1) + 3\zeta(3)^2 - \frac{3}{1120}\pi^6$$

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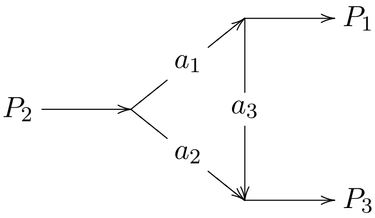
- Appear in the evaluation of Feynman diagrams:



Pizza!

THANK YOU!

- Moments of random walks:
  http://www.carma.newcastle.edu.au/~jb616/walks.pdf,
  http://www.carma.newcastle.edu.au/~jb616/walks2.pdf
- Densities of random walks:
  arXiv:1103.2995
- Mahler measures and log-sine integrals:
  arXiv:1103.3893, arXiv:1103.3035, arXiv:1103.4298

A generating function

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- Therefore:

$$\sum_{k=0}^{\infty} W_n(2k)\frac{(-x)^k}{(k!)^2} = \sum_{k=0}^{\infty}\sum_{a_1+\cdots+a_n=k}\frac{(-x)^k}{(a_1!)^2\cdots(a_n!)^2}$$

$$= \left(\sum_{a=0}^{\infty}\frac{(-x)^a}{(a!)^2}\right)^n = J_0(2\sqrt{x})^n$$

Ramanujan's Master Theorem

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For "nice" analytic functions φ,

$$\int_0^\infty x^{\nu-1} \left(\sum_{k=0}^\infty \frac{(-1)^k}{k!} \varphi(k) x^k \right) \mathrm{d}x = \Gamma(\nu)\varphi(-\nu).$$

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- Begs to be applied to

$$\sum_{k=0}^\infty W_n(2k)\frac{(-x)^k}{(k!)^2} = J_0(2\sqrt{x})^n$$

by setting $\varphi(k) = \dfrac{W_n(2k)}{k!}$

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- First and more inspiredly found by David Broadhurst building on work of J.C. Kluyver

David Broadhurst. "Bessel moments, random walks and Calabi-Yau equations." Preprint, Nov 2009.

J.C. Kluyver. "A local probability problem." *Nederl. Acad. Wetensch. Proc.*, **8**, 341–350, 1906.

A convolution formula

Conjecture

For even n,

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- True for even s
- True for $n = 2$
- True for $n = 4$ and integer s
- In general, proven up to some technical growth conditions