An application of modular forms to short random walks

Building Bridges: 1st EU-US conference on Automorphic Forms and related topics

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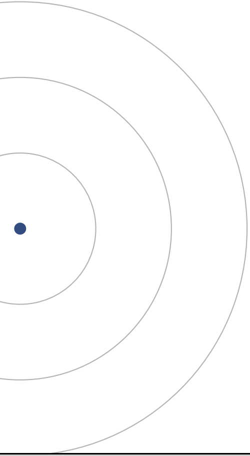
Random walks



- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

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$p_n(x)$	probability density
$W_n(s)$	sth moment

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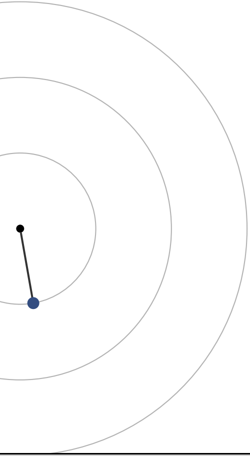

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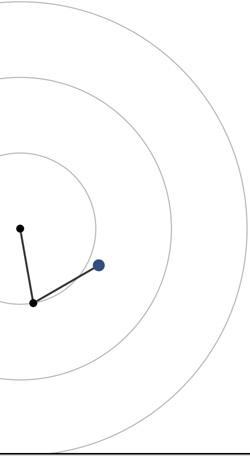

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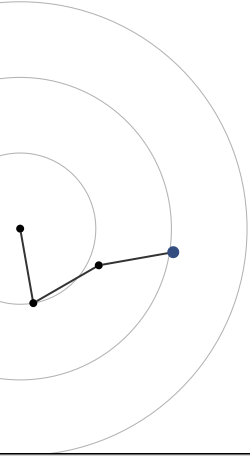

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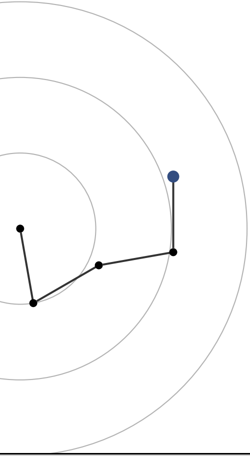

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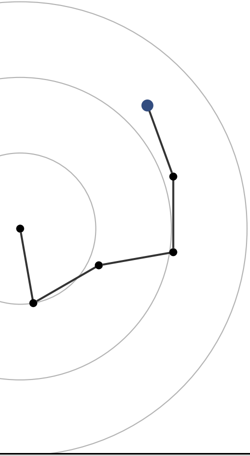

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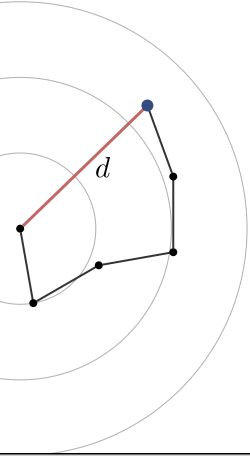

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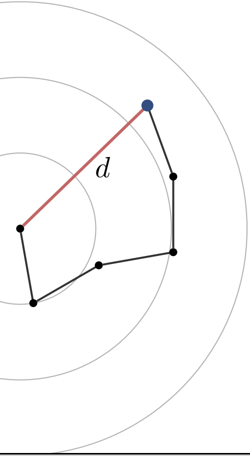

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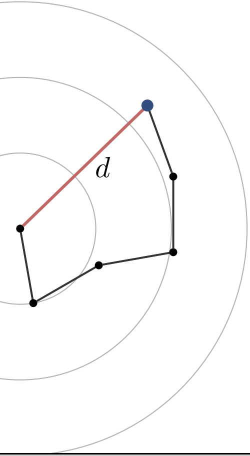
EG

$$W_2(1) = \frac{4}{\pi}$$



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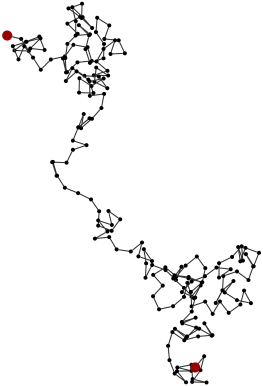
EG new

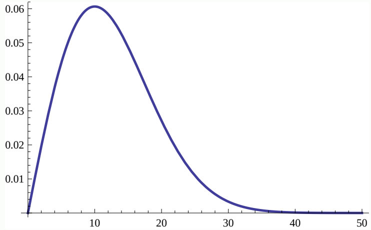
$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

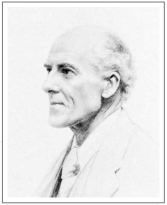


- Karl Pearson asked for $p_n(x)$ in Nature in 1905. This famous question coined the term **random walk**.
- **Asymptotic** answer by Lord Rayleigh:

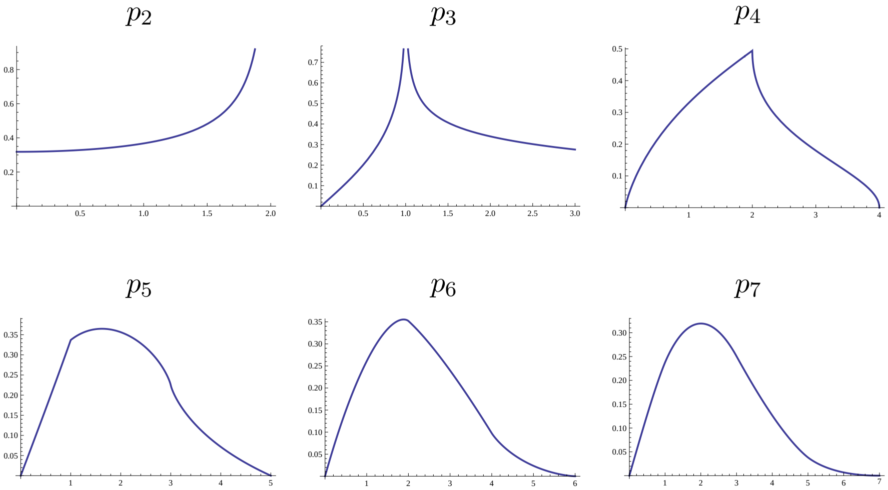
$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n}$$

EG p_{200}



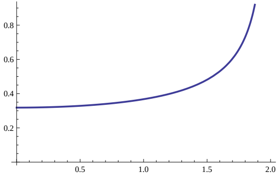


Densities of short walks

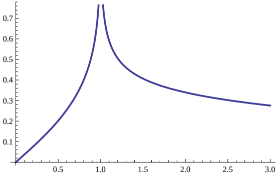


Densities of short walks

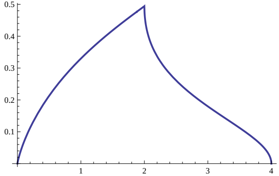
p_2



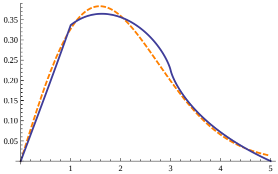
p_3



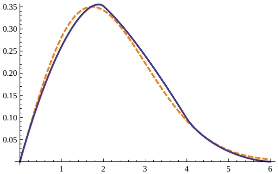
p_4



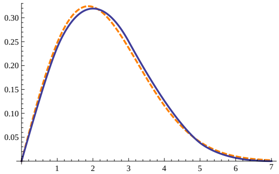
p_5



p_6



p_7



Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \;??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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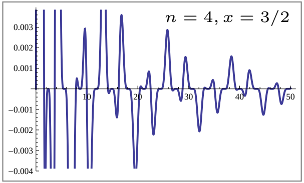

Moments of random walks

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \mathrm{d}\mathbf{x}$$

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THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

- $W_n(2k)$ counts the number of **abelian squares**: strings xy of length $2k$ from an alphabet with n letters such that y is a permutation of x.
- Introduced by Erdős and studied by others.

EG

$$W_2(2k) = \binom{2k}{k}$$

Even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$

$$W_5(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2(k-j)}{k-j} \sum_{\ell=0}^{j} \binom{j}{\ell}^2 \binom{2\ell}{\ell}$$

Complex moments

THM

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

- Inevitable **recursions** $K \cdot f(k) = f(k+1)$

$$\left[(k+2)^2 K^2 - (10k^2+30k+23)K + 9(k+1)^2\right] \cdot W_3(2k) = 0$$

$$\left[(k+2)^3 K^2 - (2k+3)(10k^2+30k+24)K + 64(k+1)^3\right] \cdot W_4(2k) = 0$$

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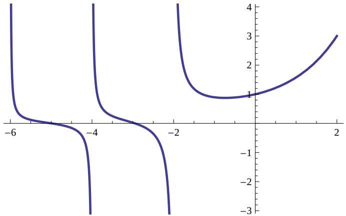
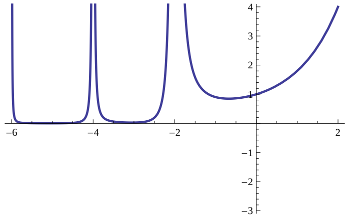
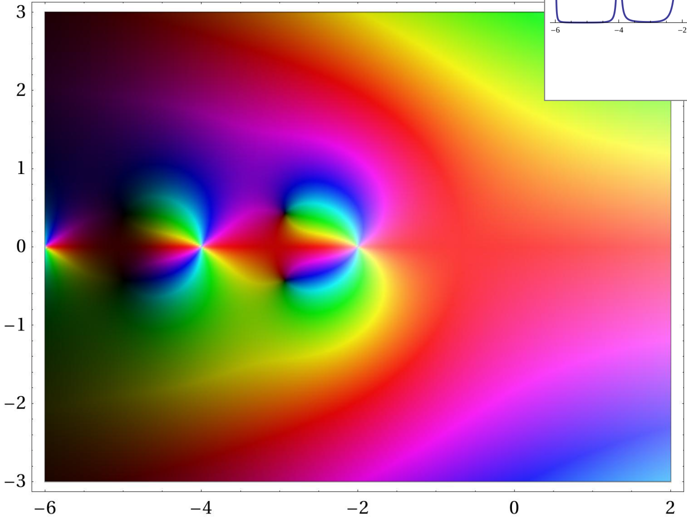
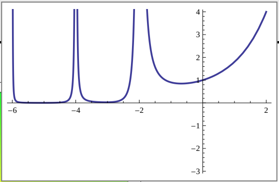
- Via **Carlson's Theorem** these become functional equations
- $W_3(s)$ has a simple pole at -2 with residue $\frac{2}{\sqrt{3}\pi}$; others at $-2k$.





$W_4(s)$ in the complex plane





Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $$\mathcal{M}\left[f;s\right] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$$

$$W_n(s-1) = \mathcal{M}\left[p_n;s\right]$$

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  $\mathcal{M}[f;s] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$
- $F(s)$ is analytic in a strip
- Functional properties:
  - $\mathcal{M}[x^\mu f(x);s] = F(s+\mu)$
  - $\mathcal{M}[D_x f(x);s] = -(s-1)F(s-1)$
  - $\mathcal{M}[-\theta_x f(x);s] = sF(s)$

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Thus functional equations for $F(s)$ translate into DEs for $f(x)$

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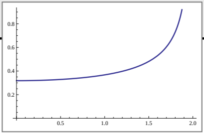
Thus functional equations for $F(s)$ translate into DEs for $f(x)$

- Poles of $F(s)$ left of strip $\implies$ asymptotics of $f(x)$ at zero

  $\frac{1}{(s+m)^{n+1}}$ $\qquad$ $\frac{(-1)^n}{n!} x^m (\log x)^n$

Mellin approach illustrated for p_2



- $W_2(2k) = \binom{2k}{k}$

$$(s+2)W_2(s+2) - 4(s+1)W_2(s) = 0$$
$$\left[x^2\left(\theta_x+1\right) - 4\theta_x\right] \cdot p_2(x) = 0$$

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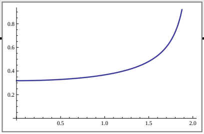

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- Hence: $p_2(x) = \frac{C}{\sqrt{4-x^2}}$

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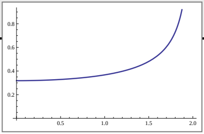

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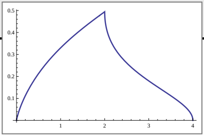
- Hence: $p_2(x) = \frac{C}{\sqrt{4-x^2}}$

$$W_2(s) = \frac{1}{\pi}\frac{1}{s+1} + O(1) \text{ as } s \to -1$$
$$p_2(x) = \frac{1}{\pi} + O(x) \text{ as } x \to 0^+$$

- Taken together: $p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$

p_4 and its differential equation



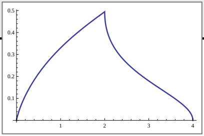
$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ with

$$A_4 = x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3$$

RK This DE is **modular**. We will come back to this shortly.

p_4 and its differential equation



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$$\begin{aligned} A_4 &= x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3 \\ &= (x-4)(x-2)x^3(x+2)(x+4)D_x^3 + 6x^4\left(x^2-10\right)D_x^2 \\ &\quad + x\left(7x^4-32x^2+64\right)D_x + \left(x^2-8\right)\left(x^2+8\right) \end{aligned}$$

RK This DE

The leading coefficient for general n always factors analogously. In light of an explicit recursion by Verrill (2004), this is embodied in the combinatorial identity

$$\sum_{\substack{0 \leqslant m_1,\ldots,m_j < n/2 \\ m_i < m_{i+1}}} \prod_{i=1}^{j} (n-2m_i)^2 = \sum_{\substack{1 \leqslant \alpha_1,\ldots,\alpha_j \leqslant n \\ \alpha_i \leqslant \alpha_{i+1}-2}} \prod_{i=1}^{j} \alpha_i(n+1-\alpha_i).$$

First proven by Djakov-Mityagin (2004). Direct combinatorial proof by Zagier.

EG

$$W_4(s) = \frac{3}{2\pi^2}\frac{1}{(s+2)^2} + \frac{9\log 2}{2\pi^2}\frac{1}{s+2} + O(1) \quad \text{as } s \to -2$$

$$p_4(x) = -\frac{3}{2\pi^2}x\log(x) + \frac{9\log 2}{2\pi^2}x + O(x^3) \quad \text{as } x \to 0^+$$

- $W_4(s)$ has double poles:

$$W_4(s) = \frac{s_{4,k}}{(s+2k+2)^2} + \frac{r_{4,k}}{s+2k+2} + O(1) \quad \text{as } s \to -2k-2$$

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p_4 and its asymptotics at zero

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☺☺

$$s_{4,k} = \frac{3}{2\pi^2}\frac{W_4(2k)}{8^{2k}} \qquad W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2n-2j}{n-j}$$

$r_{4,k}$ known recursively

Domb numbers

The Domb numbers

- $y_0(z) := \sum_{k \geqslant 0} W_4(2k) z^k$ is the analytic solution of

$$\left[64z^2(\theta+1)^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + \theta^3\right] \cdot y(z) = 0. \quad \text{(DE)}$$

- Let $y_1(z)$ solve (DE) and $y_1(z) - y_0(z)\log(z) \in z\mathbb{Q}[[z]]$.
  Then $p_4(x) = -\frac{3x}{4\pi^2}\, y_1(x^2/64)$.

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THM Chan-Chan-Liu 2004; Rogers 2009

Generating function for Domb numbers:

$$\sum_{k=0}^{\infty} W_4(2k) z^k = \frac{1}{1-4z}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{3},\frac{1}{2},\frac{2}{3}\\ 1,1\end{matrix}\right| \frac{108z^2}{(1-4z)^3}\right)$$

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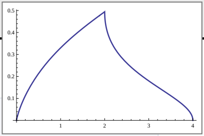
- Basis at ∞ for the hypergeometric equation of ${}_3F_2\left(\left.\begin{matrix}\frac{1}{3},\frac{1}{2},\frac{2}{3}\\ 1,1\end{matrix}\right| t\right)$:
  [as $x \to 4$ then $z = \frac{x^2}{64} \to \frac{1}{4}$ and $t = \frac{108z^2}{(1-4z)^3} \to \infty$]

$$t^{-1/3}{}_3F_2\left(\left.\begin{matrix}\frac{1}{3},\frac{1}{3},\frac{1}{3}\\ \frac{2}{3},\frac{5}{6}\end{matrix}\right| \frac{1}{t}\right), \quad t^{-1/2}{}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\right| \frac{1}{t}\right), \quad t^{-2/3}{}_3F_2\left(\left.\begin{matrix}\frac{2}{3},\frac{2}{3},\frac{2}{3}\\ \frac{4}{3},\frac{7}{6}\end{matrix}\right| \frac{1}{t}\right)$$

p_4 in hypergeometric form

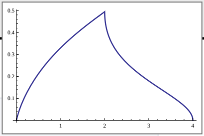


THM
Borwein-S-Wan-Zudilin 2011

For $2 \leqslant x \leqslant 4$,

$$p_4(x) = \frac{2}{\pi^2}\,\frac{\sqrt{16-x^2}}{x}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\frac{7}{6}\end{matrix}\middle|\frac{\left(16-x^2\right)^3}{108x^4}\right).$$

- Easily (if tediously) provable once found



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- Easily (if tediously) provable once found
- Quite marvelously, as first observed numerically:

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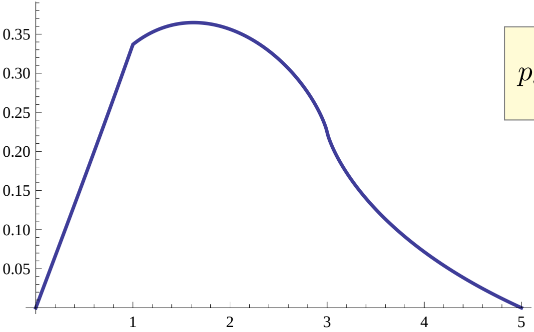
For $0 \leqslant x \leqslant 4$,

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{(16-x^2)^3}{108x^4} \right).$$

$$p_5(x) = 0.32993\,x + 0.0066167x^3 + 0.00026233x^5 + 0.000014119x^7 + O(x^9)$$



$$p_5(x) = \int_0^\infty xtJ_0(xt)J_0^5(t)\,\mathrm{d}t$$

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Karl Pearson, *1906*

p_5 — starting startlingly straight

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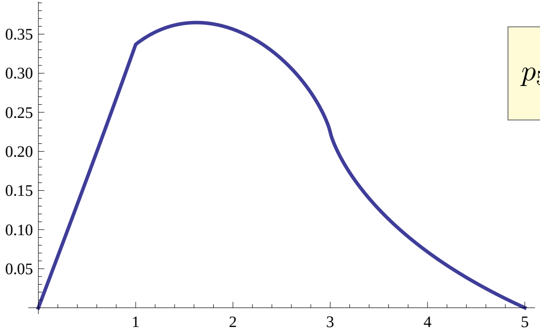

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Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
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- Dedekind eta function: $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, $\quad q = e^{2\pi i \tau}$

EG Chan-Chan-Liu 2004

$$x(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6 = -q - 6q^2 - 21q^3 - 68q^4 + \ldots, \qquad f(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2} = 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \ldots$$

Here, $\Gamma = \left\langle \Gamma_0(6), \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix} \right\rangle$.

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Here, $\Gamma = \left\langle \Gamma_0(6), \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 3 & -2 \\ 6 & -3 \end{smallmatrix} \right) \right\rangle$. Then, in a neighborhood of $i\infty$,

$$f(\tau) = y_0(x(\tau)) = \sum_{k=0}^{\infty} W_4(2k) x(\tau)^k.$$

Modular parametrization of p_4

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\Bigg(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{=\sqrt{64x(\tau)}}\Bigg) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{=\sqrt{-x(\tau)}f(\tau)}$$

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THM Borwein-S-Wan-Zudilin 2011

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- Applying the Chowla–Selberg formula, eventually leads to:

COR

$$p_4(1) = p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Chowla–Selberg formula

THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi|d|)^{6h}} \left[\prod_{k=1}^{|d|} \Gamma\left(\frac{k}{|d|}\right)^{\left(\frac{d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $d < 0$. Further, $\tau_j = \frac{-b_j + \sqrt{d}}{2a_j}$.

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EG $\mathbb{Q}(\sqrt{-15})$ has discriminant $\Delta = -15$ and class number $h = 2$.

$$Q_1 = [1, 1, 4], \qquad Q_2 = [2, 1, 2]$$

with corresponding roots

$$\tau_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{-15}, \qquad \tau_2 = \frac{1}{2}\tau_1.$$

$$\frac{1}{\sqrt{2}} |\eta(\tau_1)\eta(\tau_2)|^2 = \frac{1}{30\pi} \left(\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\Gamma(\frac{7}{15})\Gamma(\frac{11}{15})\Gamma(\frac{13}{15})\Gamma(\frac{14}{15})} \right)^{1/2}$$

$$= \frac{1}{120\pi^3} \Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15})$$

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

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  We want $g(\tau_D) = 3^6 \left(\frac{\eta(\tau_1 - 1)}{\eta(\tau_D)}\right)^{12} = -3^6 \left(\frac{\eta(\tau_1)}{\eta(\tau_D)}\right)^{12}$.

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- If $\Phi(g(\tau), g(4\tau)) = 0$, then $\Phi(g(A\tau), g(\tau)) = 0$ and $\Phi(g(N\tau), g(\tau)) = 0$.
- In particular, $\Phi(g(\tau_D), g(\tau_D)) = 0$.

Evaluating eta-quotients

EG cont.

$$\Phi(x,y) = x^5y^4 + x^4y^5 + 48\,x^5y^3 + 120\,x^4y^4 + 48\,x^3y^5 + \ldots + 387420489\,x^2y + 387420489\,xy^2$$

Modular equations such as $\Phi(g(\tau), g(4\tau)) = 0$ are automatic to prove: bound valence and test q-expansion at ∞.

Evaluating eta-quotients

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Modular equations such as $\Phi(g(\tau), g(4\tau)) = 0$ are automatic to prove: bound valence and test q-expansion at ∞.

$$\Phi(x,x) = 2\,(x-27)^2\left(x^2+81\,x+729\right)^2 x^3$$

It follows that

$$g(\tau_D) = -3^6\left(\frac{\eta(\tau_1)}{\eta(\tau_D)}\right)^{12} = -3^3\left(\frac{1+\sqrt{5}}{2}\right)^{-2}.$$

Some problems

- What is a computationally better way to evaluate generic eta quotients at quadratic irrationalities?
  Note that in the sample calculation we only needed $\Phi(x, x)$.
- Describe the double cosets $\Gamma A \Gamma$ where, e.g., $\Gamma = \Gamma_0(N)$.
  For $N = 1$ this is done by the Smith normal form.
- What more can be said about p_5?
  We know it satisfies a (non-modular) DE, as well as its expansion at zero.
  Conjecture: $p_5'''(0) = \frac{78}{225} p_5'(0) - \frac{12}{5\pi^4} \frac{1}{p_5'(0)}$

THANK YOU!

- Slides for this talk will be available from my website:
  http://arminstraub.com/talks

J. Borwein, D. Nuyens, A. Straub, J. Wan
Some arithmetic properties of short random walk integrals
The Ramanujan Journal, Vol. 26, Nr. 1, 2011, p. 109-132

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics — to appear