On the q-binomial coefficients and binomial congruences

q-series seminar

University of Illinois at Urbana–Champaign

Armin Straub

November 15, 2012

University of Illinois
at Urbana–Champaign

Our goal today

- Following a question of Andrews we seek a q-analog of:

THM Ljunggren 1952 For primes $p \geqslant 5$: $$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

George Andrews
q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher
Discrete Mathematics 204, 1999

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

- The q-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

D1

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

- The q-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

For q-series fans:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

D1

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$
$$= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= \underbrace{(1-q+q^2)}_{=\Phi_6(q)}\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q = 1$
  and hence invisible in the classical world

Cyclotomic polynomials

The nth cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

- This is an **irreducible** polynomial with **integer** coefficients.
  irreducibility due to Gauss — nontrivial

Cyclotomic polynomials

The nth cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

- This is an **irreducible** polynomial with **integer** coefficients.
  irreducibility due to Gauss — nontrivial
- $[n]_q = \dfrac{q^n - 1}{q - 1} = \displaystyle\prod_{\substack{1 < d \leqslant n \\ d|n}} \Phi_d(q)$ For primes: $[p]_q = \Phi_p(q)$

Some cyclotomic polynomials exhibited

EG

$$\begin{aligned}
\Phi_2(q) &= q+1\\
\Phi_3(q) &= q^2+q+1\\
\Phi_6(q) &= q^2-q+1\\
\Phi_9(q) &= q^6+q^3+1\\
\Phi_{21}(q) &= q^{12}-q^{11}+q^9-q^8+q^6-q^4+q^3-q+1\\
&\vdots\\
\Phi_{102}(q) &= q^{32}+q^{31}-q^{29}-q^{28}+q^{26}+q^{25}-q^{23}-q^{22}+q^{20}\\
&\quad+q^{19}-q^{17}-q^{16}-q^{15}+q^{13}+q^{12}-q^{10}-q^9+q^7\\
&\quad+q^6-q^4-q^3+q+1
\end{aligned}$$

Some cyclotomic polynomials exhibited

EG

$$\begin{aligned}
\Phi_2(q) &= q+1\\
\Phi_3(q) &= q^2+q+1\\
\Phi_6(q) &= q^2-q+1\\
\Phi_9(q) &= q^6+q^3+1\\
\Phi_{21}(q) &= q^{12}-q^{11}+q^9-q^8+q^6-q^4+q^3-q+1\\
&\vdots\\
\Phi_{105}(q) &= q^{48}+q^{47}+q^{46}-q^{43}-q^{42}-2q^{41}-q^{40}-q^{39}\\
&\quad +q^{36}+q^{35}+q^{34}+q^{33}+q^{32}+q^{31}-q^{28}-q^{26}-q^{24}\\
&\quad -q^{22}-q^{20}+q^{17}+q^{16}+q^{15}+q^{14}+q^{13}+q^{12}-q^9\\
&\quad -q^8-2q^7-q^6-q^5+q^2+q+1
\end{aligned}$$

Back to q-binomials

- $[n]_q = \dfrac{q^n - 1}{q - 1} = \prod_{\substack{1<d\leqslant n \\ d|n}} \Phi_d(q)$
- $\dbinom{n}{k}_q = \dfrac{[n]_q \, [n-1]_q \cdots [n-k+1]_q}{[k]_q \, [k-1]_q \cdots [1]_q}$
- How often does $\Phi_d(q)$ appear in this?
  - It appears $\left\lfloor \dfrac{n}{d} \right\rfloor - \left\lfloor \dfrac{n-k}{d} \right\rfloor - \left\lfloor \dfrac{k}{d} \right\rfloor$ times

Back to q-binomials

- $[n]_q = \dfrac{q^n - 1}{q - 1} = \prod_{\substack{1<d\leqslant n \\ d|n}} \Phi_d(q)$
- $\dbinom{n}{k}_q = \dfrac{[n]_q\,[n-1]_q \cdots [n-k+1]_q}{[k]_q\,[k-1]_q \cdots [1]_q}$
- How often does $\Phi_d(q)$ appear in this?
  - It appears $\left\lfloor \dfrac{n}{d} \right\rfloor - \left\lfloor \dfrac{n-k}{d} \right\rfloor - \left\lfloor \dfrac{k}{d} \right\rfloor$ times
  - Obviously nonnegative: the q-binomials are indeed **polynomials**
  - Also at most one: **square-free**
  - $\dbinom{n}{k}_q$ always contains $\Phi_n(q)$ if $0 < k < n$.
- Good way to compute q-binomials

  and even get them factorized for free

- Here's some q-binomials in **expanded** form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ + 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are **unimodal**. Sylvester, 1878

The q-binomials can be build from the q-Pascal rule:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D2

The q-binomials can be build from the q-Pascal rule:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D2

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1+q & & 1 & \\ 1 & & 1+q(1+q) & & (1+q)+q^2 & & 1 \\ & & & \vdots & & & \end{array}$$

EG

$$\binom{4}{2}_q = 1+q+q^2+q^2(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

q-binomials: combinatorial

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

D3

$w(S) =$ "normalized sum of S"

EG

$$\underbrace{\{1,2\}}_{\to 0}, \underbrace{\{1,3\}}_{\to 1}, \underbrace{\{1,4\}}_{\to 2}, \underbrace{\{2,3\}}_{\to 2}, \underbrace{\{2,4\}}_{\to 3}, \underbrace{\{3,4\}}_{\to 4}$$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

D3

$w(S) =$ "normalized sum of S"

EG

$$\underbrace{\{1,2\}}_{\to 0}, \underbrace{\{1,3\}}_{\to 1}, \underbrace{\{1,4\}}_{\to 2}, \underbrace{\{2,3\}}_{\to 2}, \underbrace{\{2,4\}}_{\to 3}, \underbrace{\{3,4\}}_{\to 4}$$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

The coefficient of q^m in $\binom{n}{k}_q$ counts the number of

- k-element subsets of n whose normalized sum is m

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

D3

$w(S) =$ "normalized sum of S"

EG

$$\underbrace{\{1,2\}}_{\to 0}, \underbrace{\{1,3\}}_{\to 1}, \underbrace{\{1,4\}}_{\to 2}, \underbrace{\{2,3\}}_{\to 2}, \underbrace{\{2,4\}}_{\to 3}, \underbrace{\{3,4\}}_{\to 4}$$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

The coefficient of q^m in $\binom{n}{k}_q$ counts the number of

- k-element subsets of n whose normalized sum is m
- partitions λ of m whose Ferrer's diagram fits in a $k \times (n-k)$ box

q-Chu-Vandermonde

Different representations make different properties apparent!

- Chu-Vandermonde: $\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$

Different representations make different properties apparent!

- Chu-Vandermonde: $\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$
- Purely from the combinatorial representation:

$$\binom{m+n}{k}_q = \sum_{S \in \binom{m+n}{k}} q^{\sum S - k(k+1)/2}$$

q-Chu-Vandermonde

Different representations make different properties apparent!

- Chu-Vandermonde: $\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$
- Purely from the combinatorial representation:

$$\binom{m+n}{k}_q = \sum_{S \in \binom{m+n}{k}} q^{\sum S - k(k+1)/2}$$

$$= \sum_j \sum_{S_1 \in \binom{m}{j}} \sum_{S_2 \in \binom{n}{k-j}} q^{\sum S_1 + \sum S_2 + (k-j)m - k(k+1)/2}$$

Different representations make different properties apparent!

- Chu-Vandermonde: $\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$
- Purely from the combinatorial representation:

$$\begin{aligned}\binom{m+n}{k}_q &= \sum_{S \in \binom{m+n}{k}} q^{\sum S - k(k+1)/2} \\ &= \sum_j \sum_{S_1 \in \binom{m}{j}} \sum_{S_2 \in \binom{n}{k-j}} q^{\sum S_1 + \sum S_2 + (k-j)m - k(k+1)/2} \\ &= \sum_j \binom{m}{j}_q \binom{n}{k-j}_q q^{(m-j)(k-j)}\end{aligned}$$

Let q be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

Let q be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

- Number of ways to choose k linearly independent vectors in $\mathbb{F}_q^n$:

$$(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$$

Let q be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

- Number of ways to choose k linearly independent vectors in $\mathbb{F}_q^n$:

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

- Hence the number of k-dim. subspaces of $\mathbb{F}_q^n$ is:

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \binom{n}{k}_q$$

Suppose $yx = qxy$ where q commutes with x, y. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

q-binomials: noncommuting variables

Suppose $yx = qxy$ where q commutes with x, y. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

EG

$$\binom{4}{2}_q x^2 y^2 = xxyy + xyxy + xyyx + yxxy + yxyx + yyxx$$
$$= (1 + q + q^2 + q^2 + q^3 + q^4) x^2 y^2$$

Suppose $yx = qxy$ where q commutes with x, y. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

EG

$$\binom{4}{2}_q x^2 y^2 = xxyy + xyxy + xyyx + yxxy + yxyx + yyxx$$
$$= (1 + q + q^2 + q^2 + q^3 + q^4) x^2 y^2$$

- Let $X \cdot f(x) = xf(x)$ and $Q \cdot f(x) = f(qx)$. Then:

$$QX \cdot f(x) = qxf(qx) = qXQ \cdot f(x)$$

It all starts with the q-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

It all starts with the q-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

$$D_q x^s = \frac{(qx)^s - x^s}{qx - x} = \frac{q^s - 1}{q - 1} x^{s-1} = [s]_q \, x^{s-1}$$

It all starts with the q-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

$$D_q x^s = \frac{(qx)^s - x^s}{qx - x} = \frac{q^s - 1}{q - 1} x^{s-1} = [s]_q \, x^{s-1}$$

- Define $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$

- $D_q e_q^x = e_q^x$
- $e_q^x \cdot e_q^y \neq e_q^{x+y}$
  unless $yx = qxy$
- $e_q^x \cdot e_{1/q}^{-x} = 1$

It all starts with the q-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

$$D_q x^s = \frac{(qx)^s - x^s}{qx - x} = \frac{q^s - 1}{q - 1} x^{s-1} = [s]_q\, x^{s-1}$$

- Define $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$
- **Homework:** Define $\cos_q(x)$, $\sin_q(x)$, ... and develop some q-trigonometry.

- $D_q e_q^x = e_q^x$
- $e_q^x \cdot e_q^y \neq e_q^{x+y}$ unless $yx = qxy$
- $e_q^x \cdot e_{1/q}^{-x} = 1$

- Formally inverting $D_q F(x) = f(x)$ gives:

$$F(x) = \int_0^x f(x) d_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

- Formally inverting $D_qF(x) = f(x)$ gives:

$$F(x) = \int_0^x f(x) d_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

THM **Fundamental theorem of q-calculus:**
Let $0 < q < 1$. Then

$$D_qF(x) = f(x).$$

$F(x)$ is the unique such function continuous at 0 with $F(0) = 0$.

Fineprint: one needs for instance that $|f(x)x^{\alpha}|$ is bounded on some $(0, a]$.

- Define the q-gamma function as

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} d_q x$$

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

- Define the q-gamma function as

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} d_q x$$

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

D6

q-beta function:

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x$$

- $B_q(t,s) = \dfrac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}$
- $B_q(t,s) = B_q(s,t)$

- Here, $(x-a)_q^n$ is defined by:

$$f(x) = \sum_{n \geqslant 0} (D_q^n f)(a) \frac{(x-a)_q^n}{[n]_q!}$$

Explicitly: $(x-a)_q^n = (x-a)(x-qa)\cdots(x-q^{n-1}a)$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

- Via a q-version of **Pascal's rule**
- **Combinatorially**, as the generating function of the element sums of k-subsets of an n-set
- **Geometrically**, as the number of k-dimensional subspaces of $\mathbb{F}_q^n$
- **Algebraically**, via a binomial theorem for noncommuting variables
- **Analytically**, via q-integral representations
- Not touched here: **quantum groups** arising in representation theory and physics

Classical binomial congruences

John Wilson (1773, Lagrange): $(p-1)! \equiv -1 \mod p$

Charles Babbage (1819): $\binom{2p-1}{p-1} \equiv 1 \mod p^2$

Joseph Wolstenholme (1862): $\binom{2p-1}{p-1} \equiv 1 \mod p^3$

James W.L. Glaisher (1900): $\binom{mp-1}{p-1} \equiv 1 \mod p^3$

Wilhelm Ljunggren (1952): $\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$

Wilson's congruence

THM Lagrange 1773 $(p-1)! \equiv -1 \mod p$

- known to Ibn al-Haytham, ca. 1000 AD
- congruence holds **if and only if** p is a prime
- not great as a practical primality test though...

> *The problem of distinguishing prime numbers from composite numbers ... is known to be one of the most important and useful in arithmetic. ... The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated.*
>
> **C. F. Gauss**, *Disquisitiones Arithmeticae, 1801*

Babbage's congruence

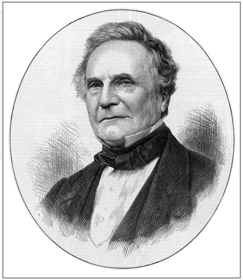
$(n-1)!+1$ is divisible by n if and only if n is a prime number

“ *In attempting to discover some analogous expression which should be divisible by n^2, whenever n is a prime, but not divisible if n is a composite number . . .* Charles Babbage is led to: ”

THM Babbage 1819

For primes $p \geqslant 3$:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$



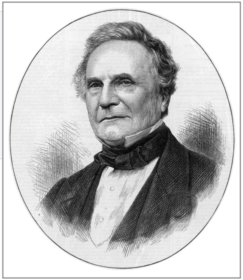
Babbage's congruence

$(n-1)!+1$ is divisible by n if and only if n is a prime number

" *In attempting to discover some analogous expression which should be divisible by n^2, whenever n is a prime, but not divisible if n is a composite number ...* Charles Babbage is led to: "

THM Babbage 1819 For primes $p \geqslant 3$:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$



- $\binom{2n-1}{n-1} = \frac{(n+1)(n+2)\cdots(2n-1)}{1\cdot 2\cdots(n-1)}$

Babbage's congruence

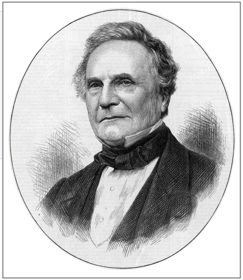
$(n-1)! + 1$ is divisible by n if and only if n is a prime number

"*In attempting to discover some analogous expression which should be divisible by n^2, whenever n is a prime, but not divisible if n is a composite number* ... Charles Babbage is led to:"

THM Babbage 1819

For primes $p \geqslant 3$:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$



$n = 16843^2$

- $\binom{2n-1}{n-1} = \frac{(n+1)(n+2)\cdots(2n-1)}{1 \cdot 2 \cdots (n-1)}$
- Does not quite characterize primes!

A simple combinatorial proof

- We have

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k}$$

- Note that p divides $\binom{p}{k}$ unless $k=0$ or $k=p$.

A simple combinatorial proof

- We have

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k}$$
$$\equiv 1 + 1 \mod p^2$$

- Note that p divides $\binom{p}{k}$ unless $k = 0$ or $k = p$.

A simple combinatorial proof

- We have

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \mod p^2$$

- Note that p divides $\binom{p}{k}$ unless $k = 0$ or $k = p$.
- $\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p}$ which is only trouble when $p = 2$

A q-analog of Babbage's congruence

- Using q-Chu-Vandermonde

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$
$$\equiv q^{p^2} + 1 \qquad \mod [p]_q^2$$

- Again, $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.

A q-analog of Babbage's congruence

- Using q-Chu-Vandermonde

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$
$$\equiv q^{p^2} + 1 \qquad \mod [p]_q^2$$

- Again, $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.

THM $$\binom{2p}{p}_q \equiv [2]_{q^{p^2}} \mod [p]_q^2$$

Extending the q-analog

- Actually, the same argument shows:

THM
Clark
1995

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2$$

Extending the q-analog

- Actually, the same argument shows:

THM Clark 1995

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2$$

- Sketch of the corresponding classical congruence:

$$\binom{ap}{bp} = \sum_{k_1+\ldots+k_a=bp} \binom{p}{k_1} \cdots \binom{p}{k_a}$$
$$\equiv \binom{a}{b} \mod p^2$$

- We get a contribution whenever b of the a many k's are p.

- Actually, the same argument shows:

No restriction on p — the argument is combinatorial.

THM Clark 1995

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2$$

- Sketch of the corresponding classical congruence:

$$\begin{aligned}\binom{ap}{bp} &= \sum_{k_1+\ldots+k_a=bp} \binom{p}{k_1} \cdots \binom{p}{k_a} \\ &\equiv \binom{a}{b} \qquad \mod p^2\end{aligned}$$

- We get a contribution whenever b of the a many k's are p.

Extending the q-analog

- Actually, the same argument shows:

No restriction on p — the argument is combinatorial.

THM Clark 1995

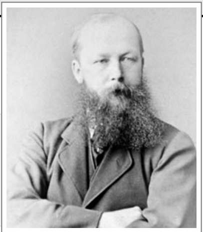
$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2$$

Similar results by Andrews; e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \mod [p]_q^2$$

George Andrews
q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher
Discrete Mathematics 204, 1999

- We get a contribution whenever b of the a many k's are p.



- Amazingly, the congruences hold modulo p^3!

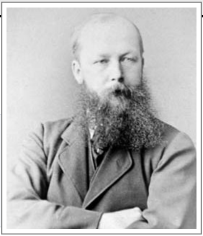
THM Wolstenholme 1862

For primes $p \geqslant 5$: $$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

> *. . . for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally . . .*

Wolstenholme and Ljunggren

- Amazingly, the congruences hold modulo p^3!



THM Wolstenholme 1862

For primes $p \geqslant 5$:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

". . . for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally . . ."

THM Ljunggren 1952

For primes $p \geqslant 5$:

$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

- Note the restriction on p — proofs are **algebraic**.

A q-analog of Ljunggren's congruence

THM
S 2011

For primes $p \geqslant 5$:

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

A q-analog of Ljunggren's congruence

THM
S 2011

For primes $p \geqslant 5$:

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

EG

Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1 + q + \ldots + q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}$ is an irreducible polynomial with integer coefficients.

THM
S 2011

For primes $p \geqslant 5$:

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

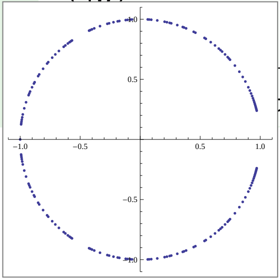
EG

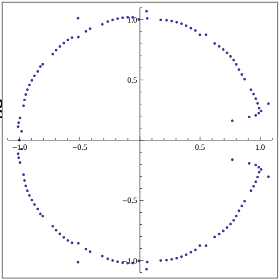
Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13} = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

— ^{32} is an irreducible

teg





Just coincidence?

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Ernst Jacobsthal (1952) proved that Ljunggren's classical congruence holds modulo p^{3+r} where r is the p-adic valuation of

$$ab(a-b)\binom{a}{b} = 2a\binom{a}{b+1}\binom{b+1}{2}.$$

- It would be interesting to see if this generalization has a nice analog in the q-world.

The case of composite numbers

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Note that $\frac{n^2-1}{12}$ is an integer if $(n,6)=1$.

The case of composite numbers

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Note that $\dfrac{n^2-1}{12}$ is an integer if $(n,6)=1$.
- Ljunggren's q-congruence holds modulo $\Phi_n(q)^3$
  over integer coefficient polynomials if $(n,6)=1$ — otherwise we get rational coefficients.

EG $n=35$, $a=2$, $b=1$

$$\binom{70}{35}_q = 1 + q^{1225} - 102(q^{35}-1)^2 + \Phi_{35}(q)^3 f(q)$$

where $f(q) = 102 + 307q + 617q^2 + \ldots + q^{1152}$

The case of composite numbers

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Note that $\dfrac{n^2-1}{12}$ is an integer if $(n,6)=1$.
- Ljunggren's q-congruence holds modulo $\Phi_n(q)^3$
  over integer coefficient polynomials if $(n,6)=1$ — otherwise we get rational coefficients.

EG $n=12$, $a=2$, $b=1$

$$\binom{24}{12}_q = 1 + q^{144} - \frac{143}{12}(q^{12}-1)^2 + \frac{1}{12}(\underbrace{1-q^2+q^4}_{\Phi_{12}(q)})^3 f(q)$$

where $f(q) = 143 + 12q + 453q^2 + \ldots + 12q^{131}$

Proof of Wolstenholme's congruence

$$\binom{2p-1}{p-1} = \frac{(2p-1)(2p-2)\cdots(p+1)}{1\cdot 2\cdots(p-1)}$$
$$= (-1)^{p-1}\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)$$

Proof of Wolstenholme's congruence

$$\binom{2p-1}{p-1} = \frac{(2p-1)(2p-2)\cdots(p+1)}{1\cdot 2\cdots(p-1)}$$
$$= (-1)^{p-1}\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)$$
$$\equiv 1 - 2p\sum_{0<i<p}\frac{1}{i} + 4p^2\sum_{0<i<j<p}\frac{1}{ij} \mod p^3$$

Proof of Wolstenholme's congruence

$$\begin{aligned}\binom{2p-1}{p-1} &= \frac{(2p-1)(2p-2)\cdots(p+1)}{1\cdot 2\cdots(p-1)}\\ &= (-1)^{p-1}\prod_{k=1}^{p-1}\left(1-\frac{2p}{k}\right)\\ &\equiv 1-2p\sum_{0<i<p}\frac{1}{i}+4p^2\sum_{0<i<j<p}\frac{1}{ij} \mod p^3\\ &= 1-2p\sum_{0<i<p}\frac{1}{i}+2p^2\left(\sum_{0<i<p}\frac{1}{i}\right)^2-2p^2\sum_{0<i<p}\frac{1}{i^2}\end{aligned}$$

Proof of Wolstenholme's congruence II

- Wolstenholme's congruence therefore follows from the fractional congruences

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2,$$

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 0 \mod p$$

Proof of Wolstenholme's congruence II

- Wolstenholme's congruence therefore follows from the fractional congruences

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2,$$

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 0 \mod p$$

EG If n is not a multiple of $p-1$ then, using a primitive root g,

$$\sum_{0<i<p} i^n \equiv \sum_{0<i<p} (gi)^n \equiv g^n \sum_{0<i<p} i^n \equiv 0 \mod p$$

Congruences for q-harmonic numbers

THM Shi-Pan 2007

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \qquad \mod [p]_q^2$$

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(q-1)^2 \qquad \mod [p]_q$$

Congruences for q-harmonic numbers

THM Shi-Pan 2007

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \qquad \bmod [p]_q^2$$

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(q-1)^2 \qquad \bmod [p]_q$$

EG $p = 5$

$$\sum_{i=1}^{4} \frac{1}{[i]_q^2} = \frac{\left(q^4+q^3+q^2+q+1\right)\left(q^6+3q^5+7q^4+9q^3+11q^2+6q+4\right)}{(q+1)^2\left(q^2+1\right)^2\left(q^2+q+1\right)^2}$$

Congruences for q-harmonic numbers

THM Shi-Pan 2007

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \qquad \bmod [p]_q^2$$

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(q-1)^2 \qquad \bmod [p]_q$$

EG $p=5$

$$\sum_{i=1}^{4} \frac{1}{[i]_q^2} = \frac{\left(q^4+q^3+q^2+q+1\right)\left(q^6+3q^5+7q^4+9q^3+11q^2+6q+4\right)}{(q+1)^2\left(q^2+1\right)^2\left(q^2+q+1\right)^2}$$

- Equivalent congruences can be given for $\displaystyle\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^n}$

  This choice actually appears a bit more natural

An exemplatory proof

- We wish to prove

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \qquad \mod [p]_q$$

Ling-Ling Shi and Hao Pan
A q-Analogue of Wolstenholme's Harmonic Series Congruence
The American Mathematical Monthly, 144(6), 2007

An exemplatory proof

- We wish to prove

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \qquad \mod [p]_q$$

- Write:

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} = (1-q)^2 \underbrace{\sum_{i=1}^{p-1} \frac{q^i}{(1-q^i)^2}}_{=:G(q)}$$

Ling-Ling Shi and Hao Pan
A q-Analogue of Wolstenholme's Harmonic Series Congruence
The American Mathematical Monthly, 144(6), 2007

An exemplatory proof

- We wish to prove

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \qquad \mod [p]_q$$

- Write:

$$[p]_q = \prod_{m=1}^{p-1} (q - \zeta^m)$$

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} = (1-q)^2 \underbrace{\sum_{i=1}^{p-1} \frac{q^i}{(1-q^i)^2}}_{=:G(q)}$$

- Hence we need to prove: $G(\zeta^m) = -\frac{p^2-1}{12}$ for $m = 1, 2, \ldots, p-1$

Ling-Ling Shi and Hao Pan
A q-Analogue of Wolstenholme's Harmonic Series Congruence
The American Mathematical Monthly, 144(6), 2007

An exemplatory proof

- We wish to prove

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \qquad \mod [p]_q$$

- Write:

$$\sum_{i=1}^{p-1} \frac{q^i}{[i]_q^2} = (1-q)^2 \underbrace{\sum_{i=1}^{p-1} \frac{q^i}{(1-q^i)^2}}_{=:G(q)}$$

$$[p]_q = \prod_{m=1}^{p-1} (q - \zeta^m)$$

- Hence we need to prove: $G(\zeta^m) = -\frac{p^2-1}{12}$ for $m = 1, 2, \ldots, p-1$
- $G(\zeta^m) = \sum_{i=1}^{p-1} \frac{\zeta^{mi}}{(1-\zeta^{mi})^2} = \sum_{i=1}^{p-1} \frac{\zeta^i}{(1-\zeta^i)^2} = G(\zeta)$

Ling-Ling Shi and Hao Pan
A q-Analogue of Wolstenholme's Harmonic Series Congruence
The American Mathematical Monthly, 144(6), 2007

An exemplatory proof II

- Define $G(q, z) = \sum_{i=1}^{p-1} \frac{q^i}{(1 - q^i z)^2}$
- We need $G(\zeta, 1) = -\frac{p^2 - 1}{12}$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta, 1) = -\frac{p^2-1}{12}$

$$G(\zeta, z) = \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k$$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta,1) = -\frac{p^2-1}{12}$

$$G(\zeta,z) = \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k$$
$$= \sum_{k=1}^{\infty} k z^{k-1} \sum_{i=1}^{p-1} \zeta^{ki}$$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta,1) = -\frac{p^2-1}{12}$

$$\sum_{i=1}^{p-1} \zeta^{ki} = \begin{cases} p-1 & \text{if } p|k \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} G(\zeta,z) &= \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k \\ &= \sum_{k=1}^{\infty} kz^{k-1} \sum_{i=1}^{p-1} \zeta^{ki} \\ &= p\sum_{k=1}^{\infty} pkz^{k-1} - \sum_{k=1}^{\infty} kz^{k-1} \end{aligned}$$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta,1) = -\frac{p^2-1}{12}$

$$\sum_{i=1}^{p-1} \zeta^{ki} = \begin{cases} p-1 & \text{if } p|k \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} G(\zeta,z) &= \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k \\ &= \sum_{k=1}^{\infty} kz^{k-1} \sum_{i=1}^{p-1} \zeta^{ki} \\ &= p\sum_{k=1}^{\infty} pkz^{k-1} - \sum_{k=1}^{\infty} kz^{k-1} \\ &= \frac{p^2 z^{p-1}}{(1-z^p)^2} - \frac{1}{(1-z)^2} \end{aligned}$$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta,1) = -\frac{p^2-1}{12}$

$$\sum_{i=1}^{p-1} \zeta^{ki} = \begin{cases} p-1 & \text{if } p|k \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} G(\zeta,z) &= \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k \\ &= \sum_{k=1}^{\infty} kz^{k-1} \sum_{i=1}^{p-1} \zeta^{ki} \\ &= p\sum_{k=1}^{\infty} pkz^{k-1} - \sum_{k=1}^{\infty} kz^{k-1} \\ &= \frac{p^2 z^{p-1}}{(1-z^p)^2} - \frac{1}{(1-z)^2} \xrightarrow{\text{as } z \to 1} -\frac{p^2-1}{12} \end{aligned}$$

An exemplatory proof II

- Define $G(q,z) = \sum_{i=1}^{p-1} \frac{q^i}{(1-q^i z)^2}$
- We need $G(\zeta,1) = -\frac{p^2-1}{12}$

$$\sum_{i=1}^{p-1} \zeta^{ki} = \begin{cases} p-1 & \text{if } p|k \\ -1 & \text{otherwise} \end{cases}$$

$$G(\zeta,z) = \sum_{i=1}^{p-1} \zeta^i \sum_{k=0}^{\infty} \zeta^{ki}(k+1)z^k$$

$$= \sum_{k=1}^{\infty} k z^{k-1} \ldots$$

$$= p \sum_{k=1}^{\infty} pk z^{k-1} - \sum_{k=1}^{\infty} k z^{k-1}$$

$$= \frac{p^2 z^{p-1}}{(1-z^p)^2} - \frac{1}{(1-z)^2} \xrightarrow{\text{as } z \to 1} -\frac{p^2-1}{12}$$

This is beautifully generalized in:

Karl Dilcher
Determinant expressions for q-harmonic congruences and degenerate Bernoulli numbers
Electronic Journal of Combinatorics 15, 2008

Can we do better than modulo p^3?

- Are there primes p such that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^4?$$

- Such primes are called **Wolstenholme primes**.
- The only two known are 16843 and 2124679.

McIntosh, 1995: up to 10^9

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Can we do better than modulo p^3?

- Are there primes p such that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^4?$$

- Such primes are called **Wolstenholme primes**.
- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9
- Infinitely many Wolstenholme primes are conjectured to exist.

  However, no primes are conjectured to exist for modulo p^5.

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Can we do better than modulo p^3?

- Are there primes p such that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^4?$$

- Such primes are called **Wolstenholme primes**.
- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9
- Infinitely many Wolstenholme primes are conjectured to exist.
  However, no primes are conjectured to exist for modulo p^5.
- Conjecturally, Wolstenholme's congruence characterizes primes:

$$\binom{2n-1}{n-1} \equiv 1 \mod n^3 \quad \Longleftrightarrow \quad n \text{ is prime}$$

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Can we do better than modulo p^3?

- Are there primes p such that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^4?$$

- Such primes are called **Wolstenholme primes**.
- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9
- Infinitely many Wolstenholme primes are conjectured to exist.
  However, no primes are conjectured to exist for modulo p^5.
- Conjecturally, Wolstenholme's congruence characterizes primes:

$$\binom{2n-1}{n-1} \equiv 1 \mod n^3 \quad \iff \quad n \text{ is prime}$$

- Any insight into these from the q-perspective??

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Some open problems

- Extension to Jacobsthal's result?
- Extension to

$$\binom{ap}{bp} \equiv \binom{a}{b} \cdot \left[1 - ab(a-b)\frac{p^3}{3}B_{p-3}\right] \mod p^4,$$

  and insight into Wolstenholme primes?
- Is there a nice q-analog for Gauss' congruence?

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2a \mod p$$

  where $p = a^2 + b^2$ and $a \equiv 1 \bmod 4$.

  Generalized to p^2 and p^3 by Chowla-Dwork-Evans (1986) and by Cosgrave-Dilcher (2010)

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

Victor Kac and Pokman Cheung
Quantum Calculus
Springer, 2002

Armin Straub
A q-analog of Ljunggren's binomial congruence
Proceedings of FPSAC, 2011