On the ubiquity of modular forms and Apéry-like numbers

Algorithmic Combinatorics Seminar
RISC (Johannes Kepler University, Linz, Austria)

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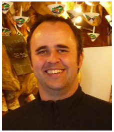
Based on joint work with:

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PART I

Encounters with Apéry numbers and modular forms

Short random walks
Binomial congruences
Positivity of rational functions
Series for $1/\pi$

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5) u_n + n^3 u_{n-1} = 0.$$

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THM **Apéry '78** $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for the Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} - (2n+1)(an^2 + an + b)u_n + cn^3 u_{n-1} = 0.$$

Q Are there other triples for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Apéry-like numbers

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Q Are there other triples for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Almkvist and Zudilin find 14 triplets (a, b, c).
  The simpler case of $(n+1)^2 u_{n+1} - (an^2 + an + b)u_n + cn^2 u_{n-1} = 0$ was similarly investigated by Beukers and Zagier.
- 4 hypergeometric, 4 Legendrian and 6 sporadic solutions

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right),\qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modular forms

> *Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist.*
>
> **Barry Mazur** (BBC Interview, "The Proof", 1997)

DEF Actions of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

- on $\tau \in \mathcal{H}$ by $\qquad \gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$,
- on $f : \mathcal{H} \to \mathbb{C}$ by $\qquad (f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma \cdot \tau)$.

EG $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$T \cdot \tau = \tau + 1, \qquad S \cdot \tau = -\frac{1}{\tau}$$

Modular forms

> *There's a saying attributed to Eichler that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.*
>
> **Andrew Wiles** (BBC Interview, "The Proof", 1997)

DEF A function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of weight k if

- $f|_k \gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,
- f is holomorphic (including at the cusp $i\infty$).

EG

$$f(\tau + 1) = f(\tau), \qquad \tau^{-k} f(-1/\tau) = f(\tau).$$

- Similarly, MFs w.r.t. finite-index $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$
- Spaces of MFs finite dimensional, Hecke operators, ...

- The **Dedekind eta function** $(q = e^{2\pi i\tau})$

$$\eta(\tau) = q^{1/24} \prod_{n \geqslant 1} (1 - q^n)$$

transforms as

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau).$$

EG $\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24}$ is a modular form of weight 12.

- For $k > 1$, the **Eisenstein series** $G_{2k}(\tau)$ is modular of weight $2k$.

$$G_{2k}(\tau) = \sum_{m,n \in \mathbb{Z}}' \frac{1}{(m\tau + n)^{2k}}$$

$$\sigma_k(n) = \sum_{d|n} d^k$$

$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{\Gamma(2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

Modular forms: Eisenstein series and L-functions

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$$\sigma_k(n) = \sum_{d|n} d^k$$

- Any modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in G_4 and G_6.

EG

$$\Delta = (60G_4)^3 - 27(140G_6)^2$$

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- Any modular form for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in G_4 and G_6.

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$$\Delta = (60G_4)^3 - 27(140G_6)^2$$

- The **L-function** of $f(\tau) = \sum_{n=0}^{\infty} b(n) q^n$ is

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} [f(i\tau) - f(i\infty)] \tau^{s-1} \mathrm{d}\tau = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

EG

$$L(G_{2k}, s) = 2\frac{(2\pi i)^{2k}}{\Gamma(2k)} \zeta(s)\zeta(s-2k+1)$$

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)}_{\text{modular function}}^{12n}.$$

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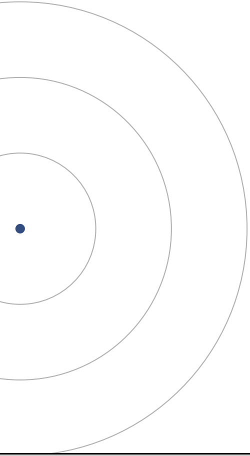
FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

Personal encounter in the wild I: Random walks



- n steps in the plane (length 1, random direction)

Personal encounter in the wild I: Random walks

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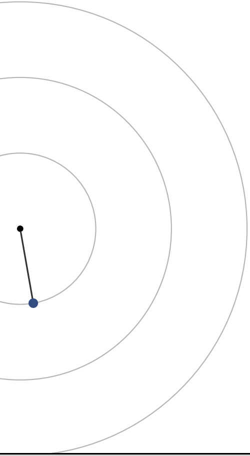

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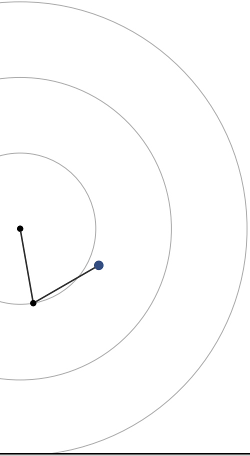

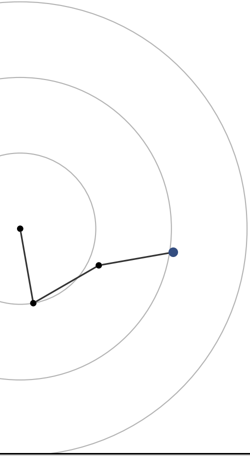

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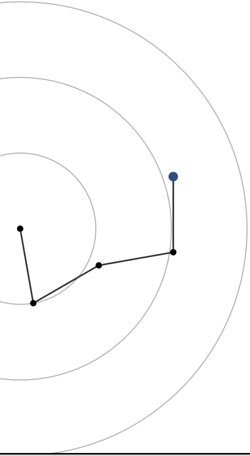

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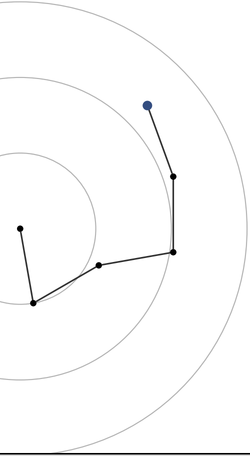

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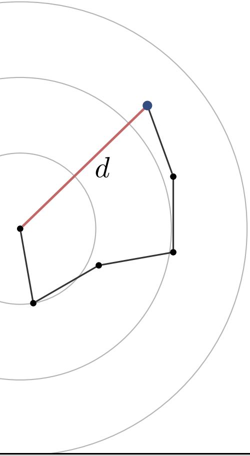
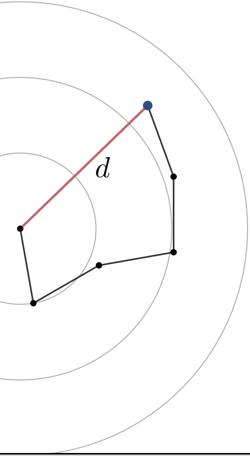
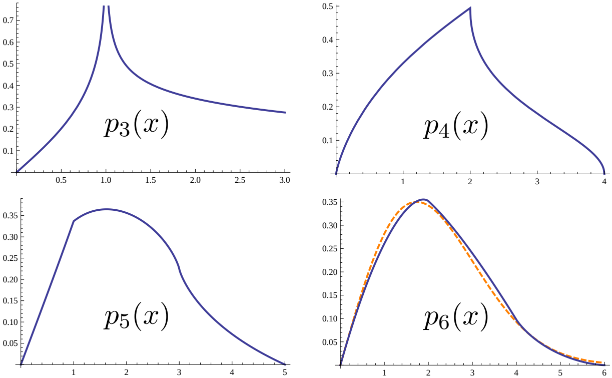


Personal encounter in the wild I: Random walks

- n steps in the plane (length 1, random direction)
- $p_n(x)$: probability density of distance traveled





- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

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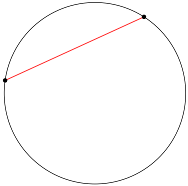
THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

Personal encounter in the wild I: Random walks

- In particular, $W_2(2k) = \binom{2k}{k}$.
- The average distance traveled in two steps is

$$W_2(1) = \binom{1}{1/2} = \frac{4}{\pi}.$$



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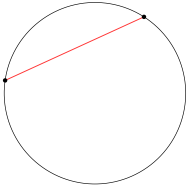

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$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, -k, -k \\ 1, 1 \end{matrix} \right| 4\right).$$

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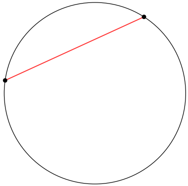

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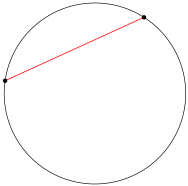
$${}_3F_2\left(\begin{matrix}\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1\end{matrix}\middle|4\right) \approx 1.574597238 - 0.126026522i$$



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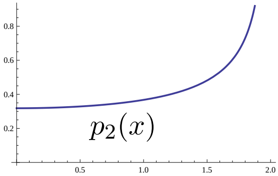
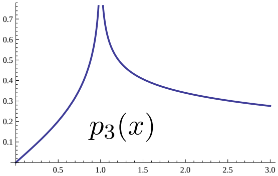
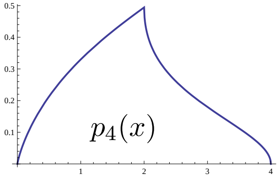
THM Borwein-Nuyens-S-Wan, 2010

$$W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6\left(\frac{2}{3}\right) = 1.57459723755189\ldots$$

Personal encounter in the wild I: Random walks







$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

easy

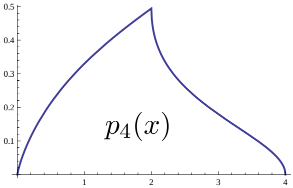
$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3} \right)$$

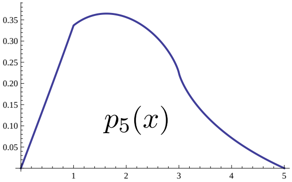
classical
with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{\left(16-x^2\right)^3}{108x^4} \right)$$

new
BSWZ 2011

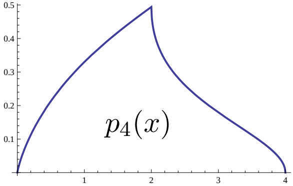
Personal encounter in the wild I: Random walks

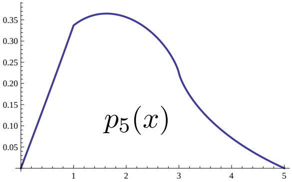




$$p_5'(0) = p_4(1) \approx 0.32993$$

Personal encounter in the wild I: Random walks





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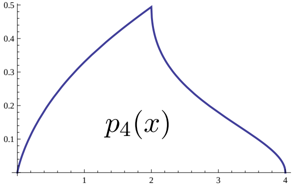
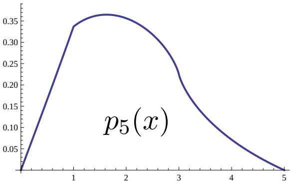
THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{\text{modular function}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{\text{modular form}}$$





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THM Borwein-S-Wan-Zudilin 2011

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- When $\tau = -\frac{1}{2} + \frac{1}{6}\sqrt{-15}$, one obtains $p_4(1)$ as an eta-product.
- Modular equations and Chowla–Selberg lead to:

$$p_4(1) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Personal encounter in the wild II: Binomial congruences

John Wilson (1773, Lagrange): $$(p-1)! \equiv -1 \mod p$$

Charles Babbage (1819): $$\binom{2p-1}{p-1} \equiv 1 \mod p^2$$

Joseph Wolstenholme (1862): $$\binom{2p-1}{p-1} \equiv 1 \mod p^3$$

James W.L. Glaisher (1900): $$\binom{mp-1}{p-1} \equiv 1 \mod p^3$$

Wilhelm Ljunggren (1952): $$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3$$

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THM
S 2011
$p \geqslant 5$

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3$$

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(pm) \equiv A(m) \mod p^3.$$

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$$A(pm) \equiv A(m) \mod p^3.$$

- The same congruence is satisfied by the **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

  Conjecturally, this extends to all Apéry-like numbers. Osburn, Sahu '09

Personal encounter in the wild II: Binomial congruences

- Wolstenholme's congruence is the $m = 1$ case of:
  The sequence $A(n) = \binom{2n}{n}$ satisfies the **supercongruence** $(p \geqslant 5)$

$$A(pm) \equiv A(m) \mod p^3.$$

- The same congruence is satisfied by the **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

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Q	How does the q-side of supercongruences for Apéry-like numbers look like?

Personal encounter in the wild III: Positivity

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The Askey–Gasper rational function $A(x,y,z)$ and the Szegő rational function $S(x,y,z)$ are positive.

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$
$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

- Both functions are on the boundary of positivity.

- WZ shows that the diagonal terms a_n of $A(x,y,z)$ satisfy

$$(n+1)^2 a_{n+1} = (7n^2+7n+2)a_n + 8n^2 a_{n-1}.$$

  This proves that they equal the **Franel numbers**

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- Using the modular parametrization of the associated Calabi–Yau differential equation, we have

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix} \middle| \frac{27z^2}{(1-2z)^3}\right).$$

- The Kauers–Zeilberger rational function

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

is conjectured to be positive.

- Its positivity implies the positivity of the Askey–Gasper function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

PROP S-Zudilin 2013 The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Personal encounter in the wild III: Positivity

- Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^d c_k e_k(x_1, \ldots, x_d)}$$

  implied by the positivity of its diagonal?
- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

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THM S-Zudilin 2013

$$h(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy}$$

is positive iff $h(x, 0)$ and the diagonal of $h(x, y)$ are positive.

Personal encounter in the wild IV: Series for $1/\pi$

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(-1)^n(4n+1)$$

- Included in first letter of Ramanujan to Hardy

  but already given by Bauer in 1859 and further studied by Glaisher

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- Included in first letter of Ramanujan to Hardy
  but already given by Bauer in 1859 and further studied by Glaisher
- Limiting case of the terminating (Zeilberger, 1994)

$$\frac{\Gamma(3/2+m)}{\Gamma(3/2)\Gamma(m+1)} = \sum_{n=0}^{\infty} \frac{(1/2)_n^2(-m)_n}{n!^2(3/2+m)_n}(-1)^n(4n+1)$$

which has a WZ proof

Carlson's theorem justifies setting $m = -1/2$.

Personal encounter in the wild IV: Series for $1/\pi$

EG
Gosper 1985

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}\frac{1103+26390n}{396^{4n}}$$

EG
Chudnovsky's 1988

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(3n)!n!^3}\frac{13591409+545140134n}{640320^{3n+3/2}}$$

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THM
Rogers-S
2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- By the first Strehl identity,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)}_{\text{modular function}}{}^n = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau).$$

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FACT

- For $\tau \in \mathbb{Q}[\sqrt{-d}]$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- The prototypical $E_2(\tau)$ satisfies

$$E_2(\tau)|_2(S-1) = \frac{6}{\pi i \tau}.$$

- These are the main ingredients for series for $1/\pi$. Mix and stir.

PART II

A secant Dirichlet series and Eichler integrals of Eisenstein series

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}$$

Secant zeta function

- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

EG
LRR '13

$$\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}, \qquad \psi_2(\sqrt{6}) = \frac{2\pi^2}{3}$$

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CONJ LRR '13 For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

Secant zeta function: Motivation

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

- Half of the Clausen and Glaisher functions reduce, e.g.,

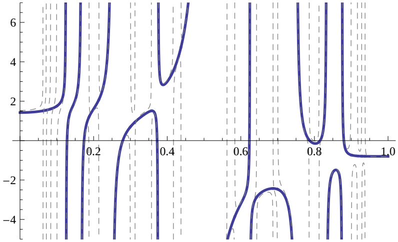
$$\sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n^{2m}} = \mathrm{poly}_m(\tau), \qquad \mathrm{poly}_1(\tau) = \frac{\tau^2}{4} - \frac{\pi\tau}{2} + \frac{\pi^2}{6}.$$

- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

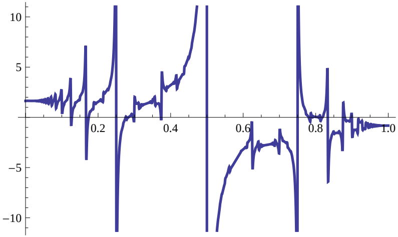
$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

In fact, this was already included in a general formula by Lerch.

- $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ has singularity at rationals with even denominator



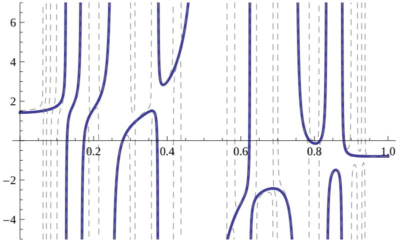
$\psi_2(\tau)$ truncated to 4 and 8 terms



Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

Secant zeta function: Convergence

- $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ has singularity at rationals with even denominator



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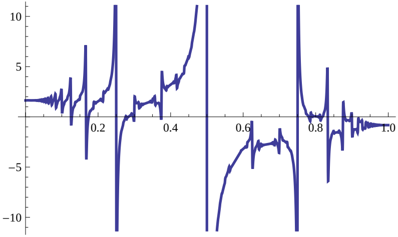

Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

THM Lalín–Rodrigue–Rogers 2013

The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Secant zeta function: Functional equation

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau + 2) = \psi_s(\tau)$.

THM LRR, BS 2013

$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

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$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\mathrm{radius}(C) \to \infty$. □

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$$= \pi^{2m}[z^{2m-1}]\frac{\sin(\tau z)}{\sin((1-\tau)z)\sin((1+\tau)z)}$$

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Secant zeta function: Fixed points

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

- In terms of the matrices A and B, the functional equations become

$$\psi_{2m}|_{1-2m}(A-1) = 0,$$
$$\psi_{2m}|_{1-2m}(B-1) = \pi^{2m} f_{2m}(\tau).$$

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- A, B, together with $-I$, generate

$$\Gamma(2) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \quad \gamma \equiv I \pmod{2}\}.$$

- Hence, for any $\gamma \in \Gamma(2)$,

$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \,\mathrm{rat}(\tau).$$

THM LRR, BS 2013

For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

proof

- As shown by Lagrange, there are X and Y which solve Pell's equation $$X^2 - rY^2 = 1.$$
- Note that $$\begin{pmatrix} X & rY \\ Y & X \end{pmatrix} \cdot \sqrt{r} = \sqrt{r}.$$

□

Secant zeta function: Special values

THM LRR, BS 2013

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proof

- As shown by Lagrange, there are X and Y which solve Pell's equation
$$X^2 - rY^2 = 1.$$
- Note that
$$\begin{pmatrix} X & rY \\ Y & X \end{pmatrix} \cdot \sqrt{r} = \sqrt{r}.$$
- Since
$$\gamma = \begin{pmatrix} X & rY \\ Y & X \end{pmatrix}^2 = \begin{pmatrix} X^2 + rY^2 & 2rXY \\ 2XY & X^2 + rY^2 \end{pmatrix} \in \Gamma(2),$$
the claim follows from the evenness of ψ_{2m} and
$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \operatorname{rat}(\tau).$$

□

Eichler integrals

THM
Bol 1949

For all sufficiently differentiable F and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$(D^{k-1}F)|_k\gamma = D^{k-1}(F|_{2-k}\gamma).$$

EG
$k = 2$

$$(DF)|_2\gamma = (c\tau+d)^{-2}F'\left(\frac{a\tau+b}{c\tau+d}\right) = D\left[F\left(\frac{a\tau+b}{c\tau+d}\right)\right]$$

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THM Bol 1949 For all sufficiently differentiable F and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

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- F is an **Eichler integral** if $D^{k-1}F$ is modular of weight k.
- Then $D^{k-1}(F|_{2-k}\gamma) = D^{k-1}F$, and hence

$$F|_{2-k}(\gamma-1) = \mathrm{poly}(\tau), \qquad \deg \mathrm{poly} \leqslant k-2.$$

- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form.

Eichler integrals

- For modular $f(\tau) = \sum a(n)q^n$, weight k, define the **Eichler integral**

$$\begin{aligned}\tilde{f}(\tau) &= \int_{\tau}^{i\infty} [f(z) - a(0)]\,(z-\tau)^{k-2}\mathrm{d}z \\ &= \frac{(-1)^k\Gamma(k-1)}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n.\end{aligned}$$

If $a(0) = 0$, $\tilde{f}$ is an Eichler integral in the strict sense of the previous slide.

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  If $a(0)=0$, $\tilde{f}$ is an Eichler integral in the strict sense of the previous slide.

- The period polynomial encodes L-values. For cusp forms f, (of level 1)

$$\tilde{f}|_{2-k}(S-1)=\int_0^{i\infty}f(z)(z-X)^{k-2}\,\mathrm{d}z$$
$$=(-1)^k\sum_{s=1}^{k-1}\binom{k-2}{s-1}\frac{\Gamma(s)}{(2\pi i)^s}L(f,s)X^{k-s-1}.$$

Eichler integrals of Eisenstein series

- The weight $2k$ Eisenstein series

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{\Gamma(2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

has Eichler integral

$$\begin{aligned} \tilde{G}_{2k}(\tau) &= \frac{4\pi i}{2k-1} \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{2k-1}} q^n \\ &= \frac{4\pi i}{2k-1} \left[\sum_{n=1}^{\infty} \frac{n^{1-2k}}{1-q^n} - \zeta(2k-1) \right]. \end{aligned}$$

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- The corresponding period "polynomial" is

$$\tilde{G}_{2k}|_{2-2k}(S-1) = \frac{(2\pi i)^{2k}}{2k-1} \left[\sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} X^{2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}} (X^{2k-2} - 1) \right].$$

Ramanujan's formula

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\alpha n}-1}\right\} = (-\beta)^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\beta n}-1}\right\}$$

$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

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THM Ramanujan, Grosswald

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$$-2^{2m} \sum_{n=0}^{m+1} (-1)^n \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} \alpha^{m-n+1} \beta^n.$$

$$\frac{1}{e^x - 1} = \frac{1}{2} \cot\left(\frac{x}{2}\right) - \frac{1}{2}$$

- In terms of

$$\xi_s(\tau) = \sum_{n=1}^{\infty} \frac{\cot(\pi n \tau)}{n^s},$$

Ramanujan's formula takes the form

$$\xi_{2k-1}|_{2-2k}(S-1) = (-1)^k (2\pi)^{2k-1} \sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} \tau^{2s-1}.$$

Secant zeta function

- Let us see that $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ is an Eichler integral as well.

  Below, $\chi_{-4} = (\frac{-4}{\cdot})$ is the nonprincipal Dirichlet character modulo 4.

LEM Berndt-S 2013

$$D^{2m}[\psi_{2m}(\tau/2)] = \frac{(2m)!}{\pi} \sum_{k,j \in \mathbb{Z}}{}' \frac{\chi_{-4}(j)}{(k\tau + j)^{2m+1}} - \frac{(-1)^m E_{2m} \pi^{2m}}{2^{2m+1}}$$

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proof Use the partial fraction expansion

$$\sec\left(\frac{\pi\tau}{2}\right) = \frac{4}{\pi} \sum_{j \geqslant 1} \frac{\chi_{-4}(j) j}{j^2 - \tau^2} = \lim_{N \to \infty} \frac{2}{\pi} \sum_{j=-N}^{N} \frac{\chi_{-4}(j)}{\tau + j},$$

and take derivatives. □

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and take derivatives. □

- $\sum_{k,j\in\mathbb{Z}}' \frac{\chi_{-4}(j)}{(k\tau+j)^{2m+1}}$ is an Eisenstein series of weight $2m+1$
  (but not for all of $\mathrm{SL}_2(\mathbb{Z})$)

Eisenstein series

- More generally, we have the Eisenstein series

$$E_k(\tau;\chi,\psi) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi(m)\psi(n)}{(m\tau+n)^k},$$

  where χ and ψ are Dirichlet characters modulo L and M.
- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

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PROP Modular transformations: $\gamma = \left(\begin{smallmatrix} a & Mb \\ Lc & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

- $E_k(\tau;\chi,\psi)|_k\gamma = \chi(d)\bar{\psi}(d)E_k(\tau;\chi,\psi)$
- $E_k(\tau;\chi,\psi)|_k S = \chi(-1)E_k(\tau;\psi,\chi)$

PROP If ψ is primitive, the L-function of $E(\tau) = E_k(\tau;\chi,\psi)$ is

$$L(E,s) = \text{const}\cdot M^s L(\chi,s)L(\bar{\psi},1-k+s).$$

Generalized Bernoulli numbers

EG Euler

$$\zeta(2n) = -\frac{1}{2}(2\pi i)^{2n}\frac{B_{2n}}{(2n)!}$$

- For integer $n > 0$ and primitive χ with $\chi(-1) = (-1)^n$,
  (χ of conductor L and Gauss sum $G(\chi)$)

$$L(n,\chi) = (-1)^{n-1}\frac{G(\chi)}{2}\left(\frac{2\pi i}{L}\right)^n \frac{B_{n,\bar{\chi}}}{n!},$$

$$L(1-n,\chi) = -B_{n,\chi}/n.$$

- The **generalized Bernoulli numbers** have generating function

$$\sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!} = \sum_{a=1}^{L}\frac{\chi(a)xe^{ax}}{e^{Lx}-1}.$$

Period polynomials of Eisenstein series

THM Berndt-S 2013

For $k \geqslant 3$ and primitive $\chi \neq 1$, $\psi \neq 1$,

$$\tilde{E}_k(X;\chi,\psi) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X;\psi,\chi) = \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}.$$

$$\text{const} = -\chi(-1)G(\chi)\,G(\psi)\frac{(2\pi i)^k}{k-1}$$

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COR Berndt-S 2013

For $k \geqslant 3$, primitive χ, $\psi \neq 1$, and n such that $L|n$,

$$R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$$

$$\tilde{E}_k(X;\chi,\psi)|_{2-k}(1-R^n) = \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}(1-(nX+1)^{k-1-s}).$$

Unimodular polynomials

DEF $p(x)$ is **unimodular** if all its zeros have absolute value 1.

- Kronecker: if $p(x) \in \mathbb{Z}[x]$ is monic and unimodular, then all nonzero roots are roots of unity.

EG Lehmer

$$x^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has only the two real roots 0.850, 1.176 off the unit circle.

EG

$$x^2 + \frac{6}{5}x + 1 = \left(x + \frac{3+4i}{5}\right)\left(x + \frac{3-4i}{5}\right)$$

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THM Cohn 1922 $P(x)$ is unimodular if and only if

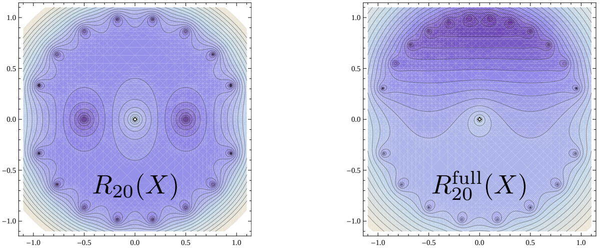
- $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ is self-inversive, i.e. $a_k = \varepsilon \overline{a_{n-k}}$ for some $|\varepsilon| = 1$, and
- $P'(x)$ has all its roots within the unit circle.

- The **Ramanujan polynomials** are

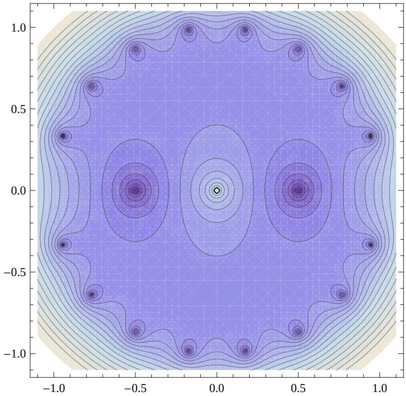
$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

THM Murty-Smyth-Wang '11: All nonreal zeros of $R_k(X)$ lie on the unit circle. For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.

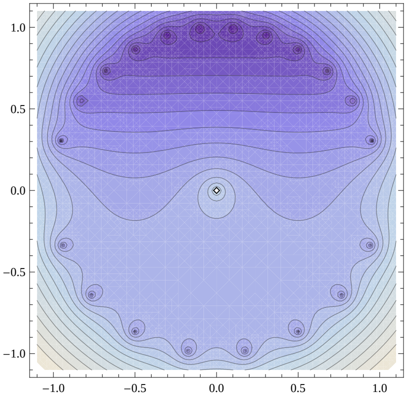
THM Lalín-Smyth '13: $R_{2k}(X) + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.



Ramanujan polynomials

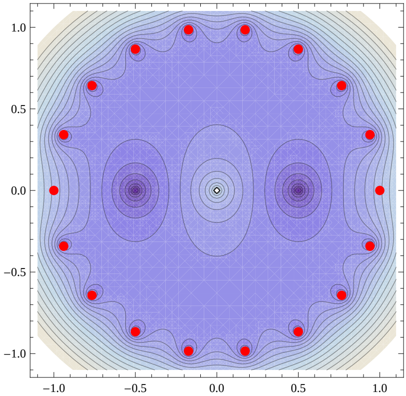


$R_{20}(X)$



$R_{20}^{\text{full}}(X)$

Ramanujan polynomials



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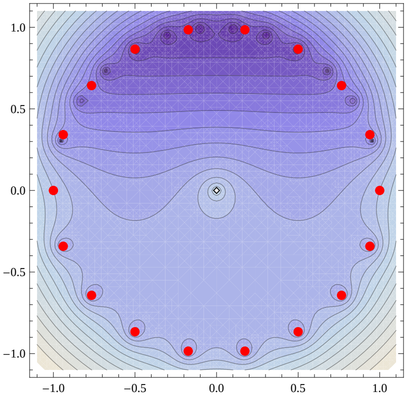

$R_{20}^{\text{full}}(X)$

Generalized Ramanujan polynomials

- We consider two kinds of **generalized Ramanujan polynomials**:

$$S_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{LX}{M}\right)^{k-s-1}$$

$$R_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M}\right)^{k-s-1} (1-X^{s-1})$$

- Obviously, $S_k(X;1,1) = R_k(X)$.

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PROP
Berndt-S
2013

- For $k > 1$, $R_{2k}(X;1,1) = R_{2k}(X)$.
- $R_k(X;\chi,\psi)$ is self-inversive.

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PROP Berndt-S 2013

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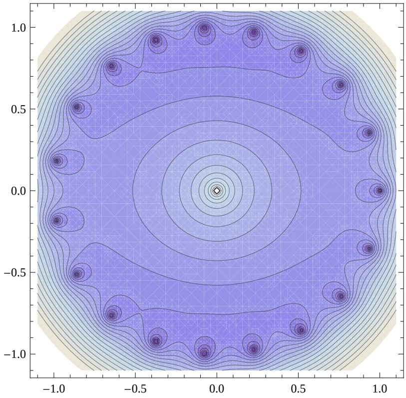
- $R_k(X;\chi,\psi)$ is self-inversive.

CONJ Berndt-S 2013

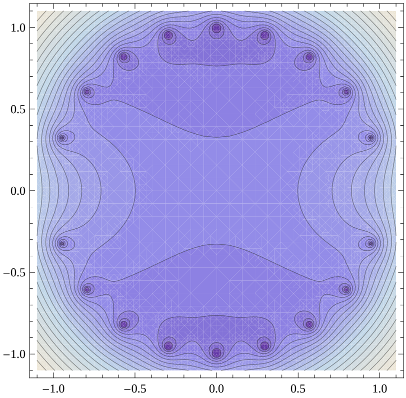
Let χ, ψ be nonprincipal real Dirichlet characters.

- $R_k(X;\chi,\psi)$ is unimodular.
- $S_k(X;\chi,\chi)$ is unimodular (up to trivial zero roots).

Generalized Ramanujan polynomials

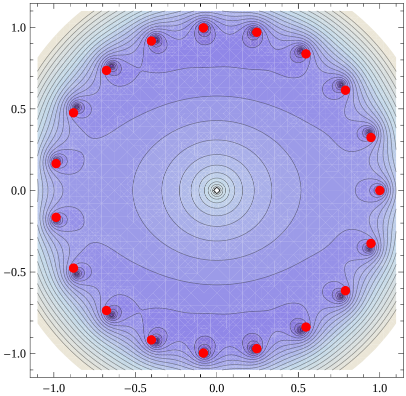


$R_{19}(X; 1, \chi_{-4})$



$S_{20}(X; \chi_{-4}, \chi_{-4})$

Generalized Ramanujan polynomials



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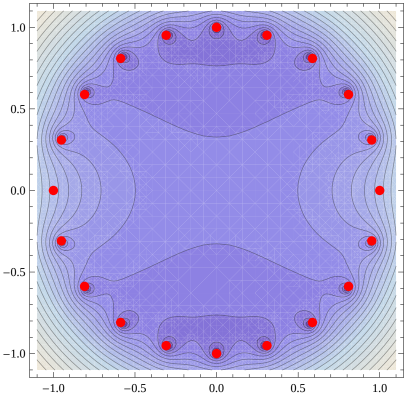

$S_{20}(X; \chi_{-4}, \chi_{-4})$

Unimodularity of period polynomials

- Both kinds of generalized Ramanujan polynomials are, essentially, period polynomials: χ, ψ primitive, nonprincipal

$$\begin{aligned} S_k(X;\chi,\psi) &= \text{const} \cdot \left[\tilde{E}_k(X;\bar{\chi},\bar{\psi}) - \psi(-1)X^{k-2}\tilde{E}_k(-1/X;\bar{\psi},\bar{\chi})\right] \\ R_k(LX+1;\chi,\psi) &= S_k(X;\chi,\psi)|_{2-k}(1-R^L) \\ &= \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L) \end{aligned}$$

Unimodularity of period polynomials

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$$R_k(LX+1;\chi,\psi) = S_k(X;\chi,\psi)|_{2-k}(1-R^L)$$

$$= \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L)$$

THM Conrey-Farmer-Imamoglu 2012: For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm \frac{1}{2}$,
- and all remaining zeros lie on the unit circle.

THM El-Guindy–Raji 2013: For any Hecke eigenform (for $\mathrm{SL}_2(\mathbb{Z})$), the full period polynomial has all zeros on the unit circle.

Application: Grosswald-type formula for Dirichlet L-values

THM Berndt-S 2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$,
($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

$$L(k-1, \chi) = \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k \left(\frac{\alpha-1}{L}; \chi, 1 \right) - \alpha^{k-2} \tilde{E}_k \left(\frac{1-1/\alpha}{L}; \chi, 1 \right) \right]$$

$$= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1-e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1-e^{2\pi i n(1/\alpha-1)/L}} \right].$$

THM Berndt-S 2013

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$$= \frac{2}{1-\alpha^{k-2}}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{k-1}}\left[\frac{1}{1-e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1-e^{2\pi i n(1/\alpha-1)/L}}\right].$$

THM Gun–Murty–Rath 2011

$$\frac{1}{\pi}\left[\tilde{E}_{2k}(\beta;1,1) - \beta^{2k-2}\tilde{E}_{2k}(-1/\beta;1,1)\right]$$

is transcendental for every algebraic $\beta \in \mathcal{H}$ with at most $2k+\delta$ exceptions. (with $\delta \in \{0,1,2,3\}$ depending on $\gcd(k,6)$)

- This number vanishes, depending on k, for $\beta = i$, $e^{\pi i/3}$, $e^{2\pi i/3}$.
  These are are the only zeros of the period polynomials which are roots of unity.
- If other exceptional β exist, then $\zeta(2k+1) \in \bar{\mathbb{Q}} + \bar{\mathbb{Q}}\pi^{2k+1}$.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2013

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
Preprint, 2013

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

A. Straub
A q*-analog of Ljunggren's binomial congruence*
DMTCS Proceedings: FPSAC 2011, p. 897-902