Arithmetic aspects of short random walks

Oberseminar Zahlentheorie, Universität zu Köln

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Based on joint work with:



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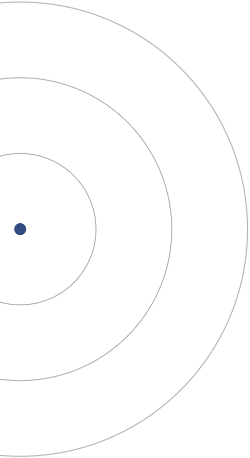
University of Newcastle, Australia

Random walks

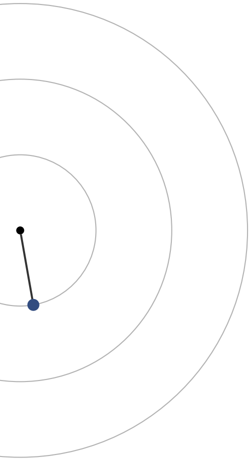


- n-step uniform planar random walk in the plane:
  - n steps, each of length 1,
  - taken in randomly chosen direction

Q What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

Random walks



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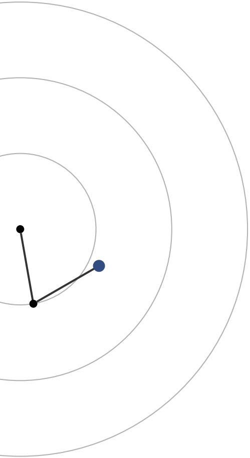

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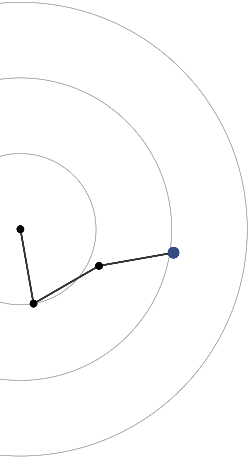

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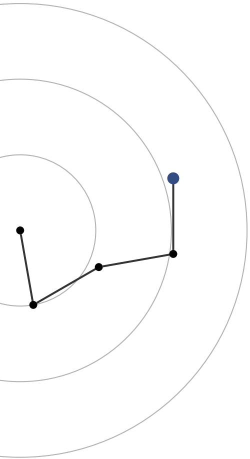

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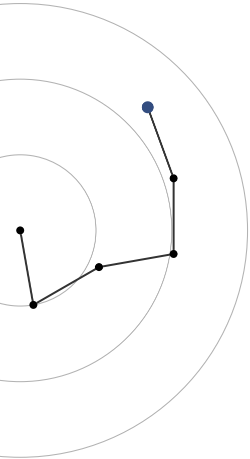

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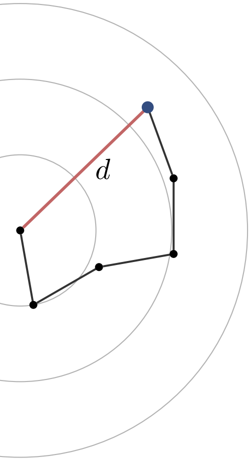

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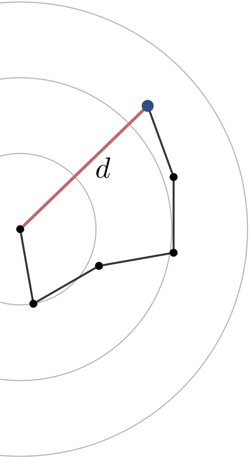

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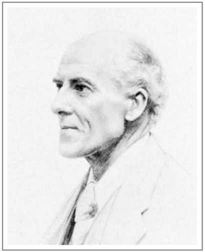
EG

$$W_2(1) = \frac{4}{\pi}$$

Random walks are only about 100 years old

- Karl Pearson asked for $p_n(x)$ in Nature in 1905.

  This famous question coined the term **random walk**.



The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

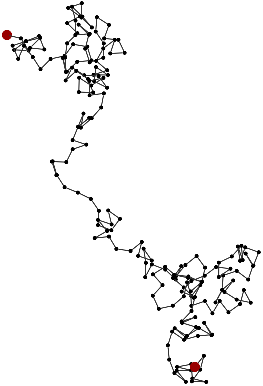
KARL PEARSON.

The Gables, East Ilsley, Berks.

Applications include:

- dispersion of mosquitoes
- random migration of micro-organisms
- phenomenon of laser speckle

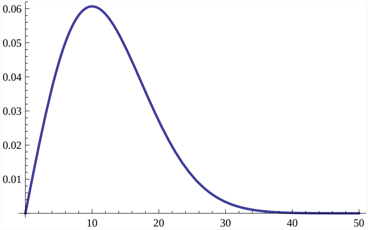
Long random walks



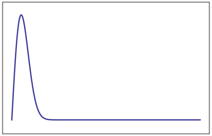
THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

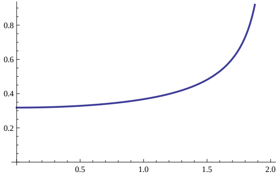
EG p_{200}



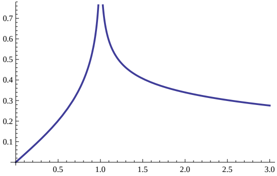
> *The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*
>
> **Karl Pearson**, *1905*



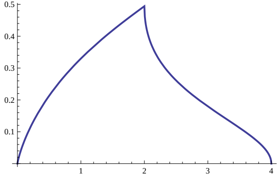
p_2



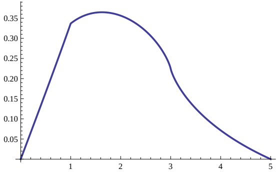
p_3



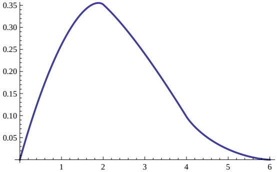
p_4



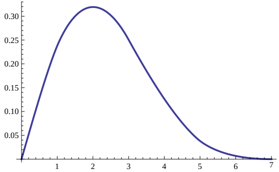
p_5

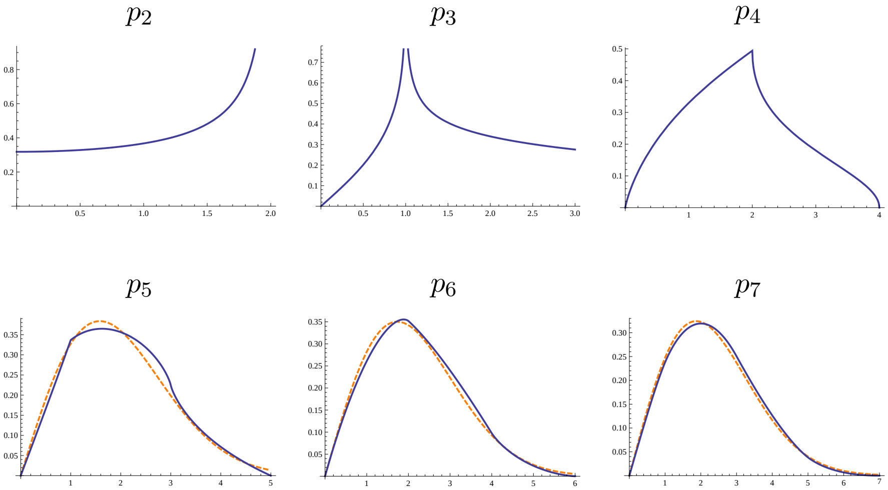


p_6

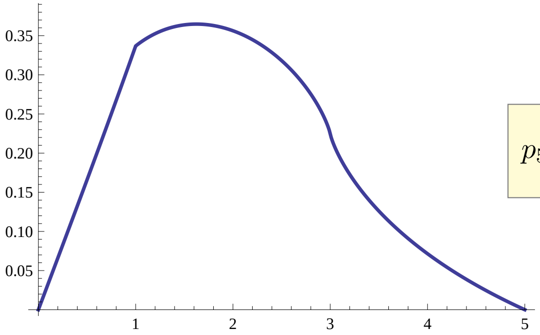


p_7




p_2
0.2 0.4 0.6 0.8
0.5 1.0 1.5 2.0
p_3
0.1 0.2 0.3 0.4 0.5 0.6 0.7
0.5 1.0 1.5 2.0 2.5 3.0
p_4
0.1 0.2 0.3 0.4 0.5
1 2 3 4
p_5
0.05 0.10 0.15 0.20 0.25 0.30 0.35
1 2 3 4 5
p_6
0.05 0.10 0.15 0.20 0.25 0.30 0.35
1 2 3 4 5 6
p_7
0.05 0.10 0.15 0.20 0.25 0.30
1 2 3 4 5 6 7

The density of a five-step random walk



$$p_5(x) = \int_0^\infty xtJ_0(xt)J_0^5(t)\,\mathrm{d}t$$

> *. . . the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a* **straight** *line. . . Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of J products to give extremely close approximations to such simple forms as horizontal lines.*
>
> **Karl Pearson**, *1906*

H. E. Fettis
On a conjecture of Karl Pearson
Rider Anniversary Volume, p. 39–54, 1963

Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \mathrm{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \;??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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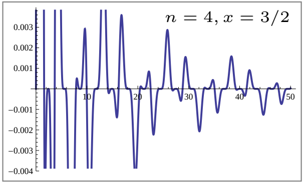

An exact probability

THM The probability that a random walk is within one unit from its origin after n steps is ...? $n > 1$

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Proof. The cumulative density function P_n can be expressed as

$$P_n(x) = \int_0^\infty x J_1(xt) J_0^n(t) \, \mathrm{d}t.$$

Then:

$$P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}.$$

□

- Recently: remarkably short proof by Olivier Bernardi

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

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$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

$$= \int_0^1 2\cos(\pi y) \mathrm{d}y$$

$$\left|1 + e^{2\pi i y}\right|$$
$$= \left|1 + (\cos \pi y + i \sin \pi y)^2\right|$$
$$= 2\cos(\pi y)$$

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$$\begin{aligned} W_2(1) &= \int_0^1 \int_0^1 \left|e^{2\pi i x} + e^{2\pi i y}\right| \mathrm{d}x\mathrm{d}y = \ ? \\ &= \int_0^1 \left|1 + e^{2\pi i y}\right| \mathrm{d}y \\ &= \int_0^1 2\cos(\pi y)\mathrm{d}y \\ &= \frac{4}{\pi} \approx 1.27324 \end{aligned}$$

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  Better: Mathematica 8 and 9 just don't evaluate the double integral.

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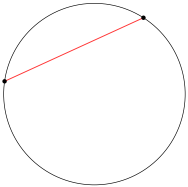
- This is the average length of a random arc on a unit circle.



Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

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- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

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$$W_5(1) \approx 2.0081618$$

- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

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$W_3(1) = 1.57459723755189\ldots = ?$

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Even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

Apéry-like

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$

Domb numbers

A combinatorial formula for the even moments

- sth moment $W_n(s)$ of the density p_n:

$$W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

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THM Borwein-Nuyens-S-Wan 2010

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2$$

- $W_n(2k)$ counts the number of **abelian squares**: strings xy of length $2k$ from an alphabet with n letters such that y is a permutation of x.
- Introduced by Erdős and studied by others.

EG $acbc\ ccba$ is an abelian square. It contributes to $W_3(8)$.

L. B. Richmond and J. Shallit
Counting abelian squares
The Electronic Journal of Combinatorics, Vol. 16, 2009.

Moments of a two-step walk

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- $W_n(s)$ is nice!
- Indeed, $W_2(s) = \binom{s}{s/2}$.

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EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} = {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -k, -k\\ 1, 1\end{matrix}\right| 4\right)$$

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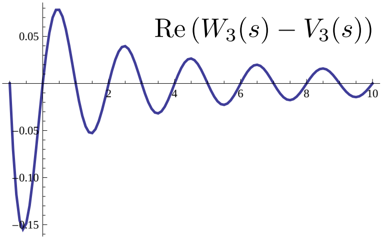

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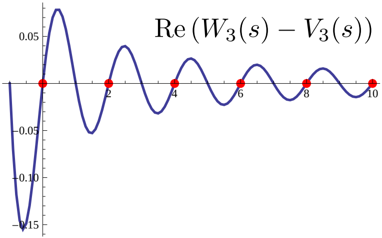

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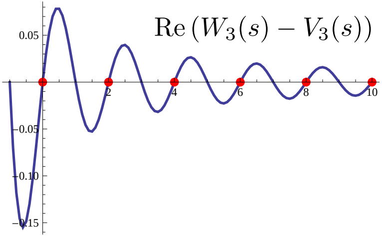
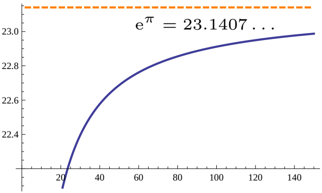


$|V_3(-i(s+1))\ /\ V_3(-is)|$:



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THM Borwein-Nuyens-S-Wan, 2010

For integers k,

$$W_3(k) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, -\frac{k}{2}, -\frac{k}{2}\\ 1, 1\end{matrix}\right| 4\right).$$

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COR

$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$
$$= 1.57459723755189\ldots$$

Moments of a four-step walk

- Using Meijer G-function representations and transformations:

THM Borwein-S-Wan, 2010

$$W_4(-1) = \frac{\pi}{4}\,{}_7F_6\left(\begin{matrix}\frac{5}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{1}{4},1,1,1,1,1\end{matrix}\middle|1\right) = \frac{\pi}{4}\,{}_6F_5\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1,1,1,1\end{matrix}\middle|1\right) + \frac{\pi}{64}\,{}_6F_5\left(\begin{matrix}\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\\ 2,2,2,2,2\end{matrix}\middle|1\right) = \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}.$$

THM Borwein-S-Wan, 2010

$$W_4(1) = \frac{3\pi}{4}\,{}_7F_6\left(\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,1,1\end{matrix}\middle|1\right) - \frac{3\pi}{8}\,{}_7F_6\left(\begin{matrix}\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},2,2,2,2,1\end{matrix}\middle|1\right).$$

- We have no idea about the case of five steps.

A combinatorial convolution

- From the interpretation as counting abelian squares:

$$W_{n+m}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_n(2j)\, W_m(2(k-j)).$$

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- True for even s
- True for $n = 2$
- True for $n = 4$ and integer s
- In general, proven up to some technical growth conditions

Complex moments

THM

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

- Inevitable **recursions**

$K \cdot f(k) = f(k+1)$

$$\left[(k+2)^2K^2 - (10k^2+30k+23)K + 9(k+1)^2\right] \cdot W_3(2k) = 0$$

$$\left[(k+2)^3K^2 - (2k+3)(10k^2+30k+24)K + 64(k+1)^3\right] \cdot W_4(2k) = 0$$

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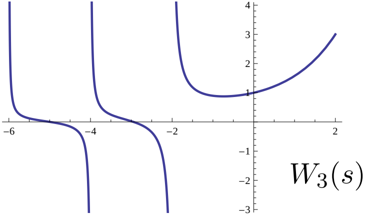
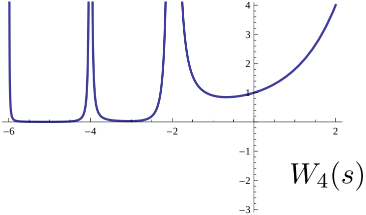
- Via **Carlson's Theorem** these become functional equations

- Analytic continuations:





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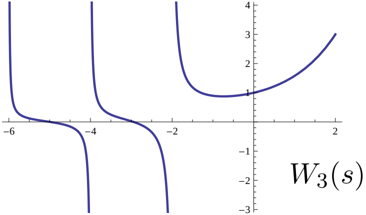

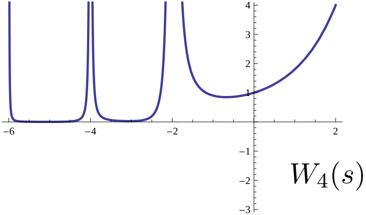

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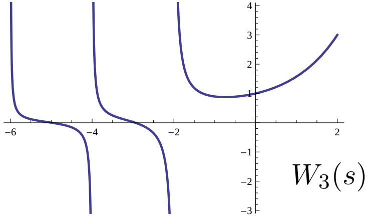

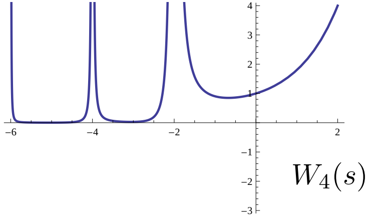

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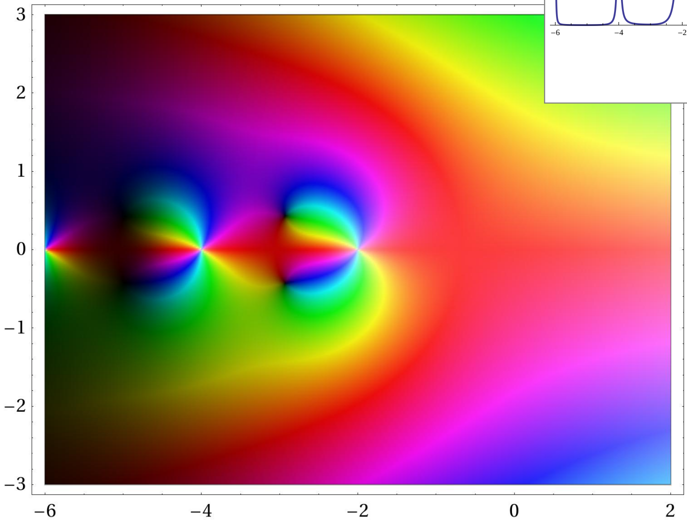
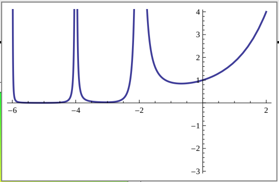
$$\frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}}$$

☺☺ $W_4(s)$ has double poles at $-2k-2$ with lowest-order term

$$\frac{3}{2\pi^2} \frac{W_4(2k)}{8^{2k}}$$

$W_4(s)$ in the complex plane





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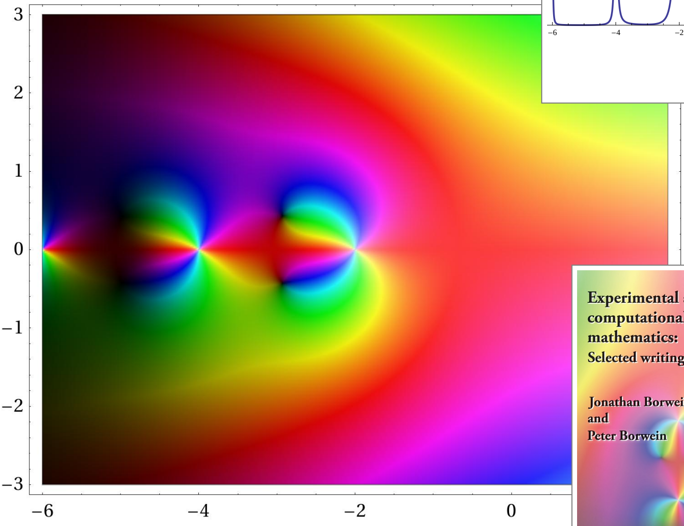

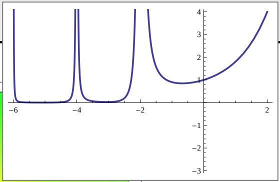

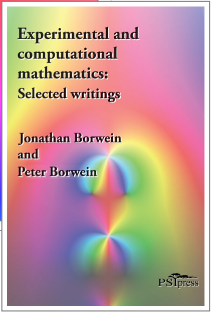

Crashcourse on the Mellin transform

- Mellin transform $F(s)$ of $f(x)$:
  $\mathcal{M}\left[f;s\right] = \int_0^\infty x^s f(x) \frac{\mathrm{d}x}{x}$

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- Functional properties:
  - $\mathcal{M}[x^\mu f(x); s] = F(s+\mu)$
  - $\mathcal{M}[D_x f(x); s] = -(s-1)F(s-1)$
  - $\mathcal{M}[-\theta_x f(x); s] = sF(s)$

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Thus functional equations for $F(s)$ translate into DEs for $f(x)$

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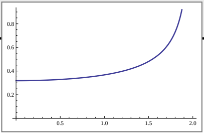
Thus functional equations for $F(s)$ translate into DEs for $f(x)$

- Poles of $F(s)$ left of strip $\Longrightarrow$ asymptotics of $f(x)$ at zero

  $\frac{1}{(s+m)^{n+1}}$ $\qquad$ $\frac{(-1)^n}{n!}x^m(\log x)^n$

Mellin approach illustrated for p_2



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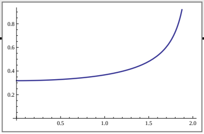

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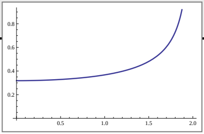

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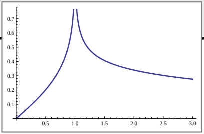
- Hence: $p_2(x) = \frac{C}{\sqrt{4-x^2}}$

$$W_2(s) = \frac{1}{\pi}\frac{1}{s+1} + O(1) \text{ as } s \to -1$$
$$p_2(x) = \frac{1}{\pi} + O(x) \text{ as } x \to 0^+$$

- Taken together: $p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$

p_3 in hypergeometric form



- $W_3(s)$ has simple poles at $-2k-2$ with residue

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$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}$$ for $0 \leqslant x \leqslant 1$

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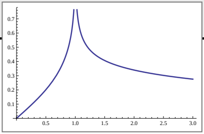

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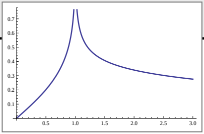

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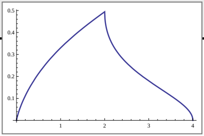
$$p_3(x) = \frac{2\sqrt{3}x}{\pi\,(3+x^2)}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{x^2\,(9-x^2)^2}{(3+x^2)^3}\right)$$

- Easy to verify once found
- Holds for $0 \leqslant x \leqslant 3$

p_4 and its differential equation



$$\left[(s+4)^3 S^4 - 4(s+3)(5s^2+30s+48)S^2 + 64(s+2)^3\right] \cdot W_4(s) = 0$$

translates into $A_4 \cdot p_4(x) = 0$ with

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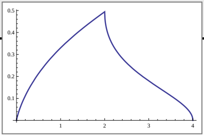

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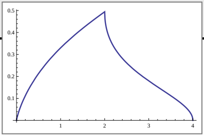

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$$\begin{aligned} A_4 &= x^4(\theta_x+1)^3 - 4x^2\theta_x(5\theta_x^2+3) + 64(\theta_x-1)^3 \\ &= (x-4)(x-2)x^3(x+2)(x+4)D_x^3 + 6x^4\left(x^2-10\right)D_x^2 \\ &\quad + x\left(7x^4-32x^2+64\right)D_x + \left(x^2-8\right)\left(x^2+8\right) \end{aligned}$$

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Densities in general

THM
Borwein-S-Wan-Zudilin, 2011

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- p_n is real analytic except at 0 and the integers $n, n-2, n-4, \ldots$.

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$$\sum_{\substack{0 \leqslant m_1, \ldots, m_j < n/2 \\ m_i < m_{i+1}}} \prod_{i=1}^{j} (n - 2m_i)^2 = \sum_{\substack{1 \leqslant \alpha_1, \ldots, \alpha_j \leqslant n \\ \alpha_i \leqslant \alpha_{i+1} - 2}} \prod_{i=1}^{j} \alpha_i (n + 1 - \alpha_i).$$

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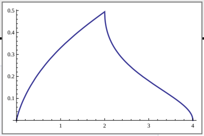

THM Chan-Chan-Liu 2004; Rogers 2009

Generating function for Domb numbers:

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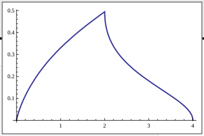
Hypergeometric forms



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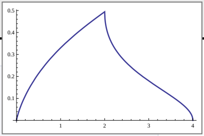

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THM Borwein-S-Wan-Zudilin 2011

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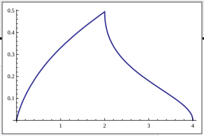

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The density of a five-step random walk, again

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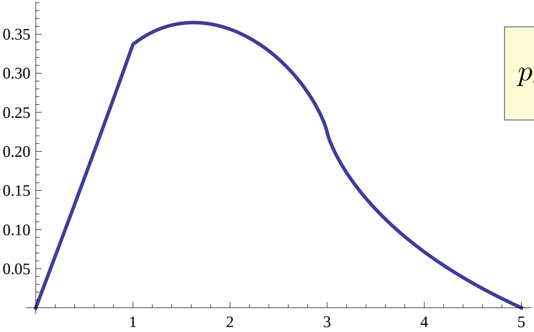

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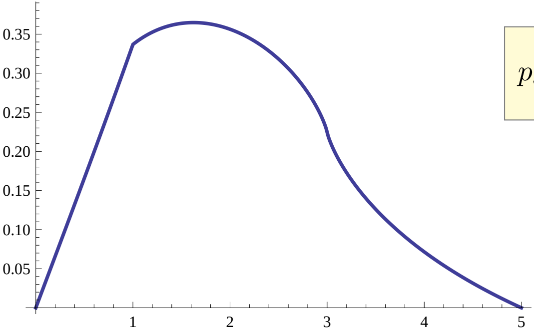

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Modular differential equations

THM Let $f(\tau)$ be a modular form and $x(\tau)$ a modular function w.r.t. Γ.

- Then $y(x)$ defined by $f(\tau) = y(x(\tau))$ satisfies a linear DE.
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EG **Classic**

$$ {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda(\tau)\right) = \theta_3(\tau)^2 $$

- $\lambda(\tau) = 16\frac{\eta(\tau/2)^8\eta(2\tau)^{16}}{\eta(\tau)^{24}}$ is the elliptic lambda function, a Hauptmodul for $\Gamma(2)$.
- $\theta_3(\tau) = \frac{\eta(\tau)^5}{\eta(\tau/2)^2\eta(2\tau)^2}$ is the usual Jacobi theta function.

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EG Chan-Chan-Liu 2004

$$x(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6, \qquad f(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}$$

$$= -q - 6q^2 - 21q^3 - 68q^4 + \ldots \qquad = 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \ldots$$

Here, $\Gamma = \left\langle \Gamma_0(6), \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix} \right\rangle$.

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$$f(\tau) = y_0(x(\tau)) = \sum_{k \geqslant 0} W_4(2k) x(\tau)^k.$$

Modular parametrization of p_4

THM Borwein-S-Wan-Zudilin 2011

For $\tau = -1/2 + iy$ and $y > 0$:

$$p_4\left(\underbrace{8i\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^3}_{=\sqrt{64x(\tau)}}\right) = \frac{6(2\tau+1)}{\pi}\underbrace{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}_{=\sqrt{-x(\tau)}f(\tau)}$$

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COR

$$p_4(1) = p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

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Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

Chowla–Selberg formula

THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi|d|)^{6h}} \left[\prod_{k=1}^{|d|} \Gamma\left(\frac{k}{|d|}\right)^{\left(\frac{d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $d < 0$. $\tau_j = \frac{-b_j + \sqrt{d}}{2a_j}$

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EG $\mathbb{Q}(\sqrt{-15})$ has discriminant $\Delta = -15$ and class number $h = 2$.

$$Q_1 = [1,1,4] \qquad Q_2 = [2,1,2]$$

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$$\frac{1}{\sqrt{2}}\left|\eta(\tau_1)\eta(\tau_2)\right|^2 = \frac{1}{30\pi}\left(\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\Gamma(\frac{7}{15})\Gamma(\frac{11}{15})\Gamma(\frac{13}{15})\Gamma(\frac{14}{15})}\right)^{1/2}$$

$$= \frac{1}{120\pi^3}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15})$$

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Proof.

- We can write $\sigma_2 = M \cdot \sigma_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ is a modular function.

□

Evaluating eta-quotients

Fact If $\sigma_1, \sigma_2 \in \mathcal{H}$ both belong to $\mathbb{Q}(\sqrt{-d})$, then the quotient $\eta(\sigma_1)/\eta(\sigma_2)$ is an algebraic number.

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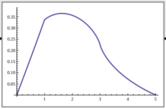
- There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.
- Then: $\Phi(f(\sigma_1), f(\sigma_1)) = 0$

□

What we know about p_5



- $W_5(s)$ has simple poles at $-2k-2$ with residue $r_{5,k}$
- Hence: $p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$

THM Borwein-S-Wan-Zudilin, 2011

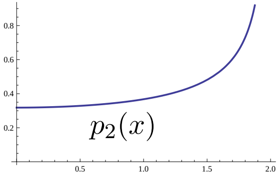
Surprising bonus of the modularity of p_4:

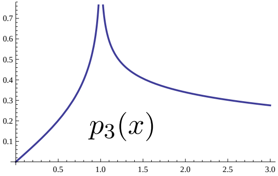
$$r_{5,0} = p_4(1) = \frac{\sqrt{5}}{40} \frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\pi^4}$$

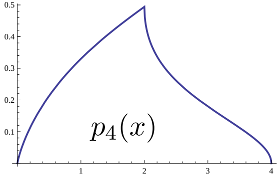
$$r_{5,1} \stackrel{?}{=} \frac{13}{225} r_{5,0} - \frac{2}{5\pi^4} \frac{1}{r_{5,0}}$$

- Other residues given recursively
- p_5 solves the DE

$$\Big[x^6(\theta+1)^4 - x^4(35\theta^4 + 42\theta^2 + 3) + x^2(259(\theta-1)^4 + 104(\theta-1)^2) - (15(\theta-3)(\theta-1))^2\Big] \cdot p_5(x) = 0$$







$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix} \middle| \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$ classical with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6}\end{matrix} \middle| \frac{\left(16-x^2\right)^3}{108x^4}\right)$$ new BSWZ

Mahler measure and random walks

DEF (Logarithmic) Mahler measure of $p(x_1, \ldots, x_n)$:

$$\mu(p) := \int_0^1 \cdots \int_0^1 \log \left| p\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$$

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- $W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i t_1} + \ldots + e^{2\pi i t_n} \right|^s \mathrm{d}\boldsymbol{t}$

EG

$$W_n'(0) = \mu(x_1 + \ldots + x_n) = \mu(1 + x_1 + \ldots + x_{n-1})$$

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EG Smyth, 1981

$$\mu(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \qquad = W_3'(0)$$

$$\mu(1 + x + y + z) = \frac{7}{2} \frac{\zeta(3)}{\pi^2} \qquad = W_4'(0)$$

$L(\chi_{-3}, s) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \ldots$

Mahler measure and random walks

EG Rogers–Zudilin, 2011

Typical conjecture (Deninger, 1997):

$$\mu(1 + x + y + 1/x + 1/y) = \left(\frac{\sqrt{-15}}{2\pi i}\right)^2 L(f_{15}, 2) = L'(f_{15}, 0)$$

where f_{15} is associated with an elliptic curve of conductor 15.

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CONJ Rodriguez-Villegas

$$W_5'(0) \overset{?}{=} \left(\frac{\sqrt{-15}}{2\pi i}\right)^5 3!\, L(g_{15},4) = -L'(g_{15},-1)$$

where $g_{15} = \eta(3\tau)^3\eta(5\tau)^3 + \eta(\tau)^3\eta(15\tau)^3$ (weight 3, level 15).

EG Rogers–Zudilin, 2011

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CONJ Rodriguez-Villegas

$$W_6'(0) \overset{?}{=} 8\left(\frac{\sqrt{-6}}{2\pi i}\right)^6 4!\, L(g_6,5) = -8L'(g_6,-1)$$

where $g_6 = \eta(\tau)^2\eta(2\tau)^2\eta(3\tau)^2\eta(6\tau)^2$ (weight 4, level 6).

Mahler measure progress

EG Chan-Chan-Liu 2004

$$x(\tau) = -\left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6, \qquad f(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}$$

$$= -q - 6q^2 - 21q^3 - 68q^4 + \ldots \qquad = 1 - 4q + 4q^2 - 4q^3 + 20q^4 + \ldots$$

$$f(\tau) = y_0(x(\tau)) = \sum_{k \geqslant 0} W_4(2k) x(\tau)^k.$$

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- Double L-function: $f = \sum a_n q^n$, $g = \sum b_n q^n$

$$L(f, g, s, t) \text{ “=” } \sum_{n \geqslant 1} \sum_{m \geqslant 0} \frac{a_n b_m}{n^s (n+m)^t}$$

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  $f = \sum a_n q^n$, $g = \sum b_n q^n$

THM
Shinder-Vlasenko 2012

$$W_5'(0) - \frac{4}{5}W_4'(0) = \frac{3\sqrt{5}\Omega_{15}^2}{20\pi}L(g_3,g_1,3,1) - \frac{3\sqrt{5}}{10\pi^3\Omega_{15}^2}L(g_2,g_1,3,1)$$

where $g_1 = \frac{Dx}{x}f$, $\quad g_2 = \frac{x}{1-x}g_1$, $\quad g_3 = \frac{x(212x^2+251x-13)}{(1-x)^3}g_1$.

The Shinder-Vlasenko starting point

- $W_n(s) = \int_{[0,1]^n} \left| e^{2\pi i t_1} + \ldots + e^{2\pi i t_n} \right|^s \mathrm{d}\boldsymbol{t}$
- $W_n'(0) = \frac{1}{2}\mu(p_n)$ where $p_n = (1 + x_1 + \ldots + x_{n-1})(1 + \frac{1}{x_1} + \ldots + \frac{1}{x_{n-1}})$

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Trick
Rodriguez-Villegas

$$\int_{[0,1]^n} \log \left[p_n\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) - \frac{1}{\lambda} \right] \mathrm{d}\boldsymbol{t}$$

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Hence, analytically continuing along the negative real axis,

$$\mu(p_n) = -\operatorname{Re} \left[\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \right]^{-1} \sum_{k \geqslant 0} W_n(2k) \lambda^k \bigg|_{\lambda = \infty}.$$

Questions and problems

- The differential equations for $n \geqslant 5$ are not modular.

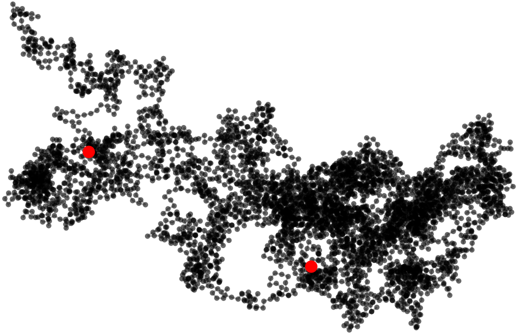
  Can one profitably bring vector-valued modular forms into the picture?
- Given a linear differential equation **automatically** find its "hypergeometric-type" solutions.

  Promising work by Mark van Hoeij and his group
- More about the five step case? Average distance travelled?

  $W_n(1) = n \int_0^\infty J_1(x) J_0(x)^{n-1} \frac{\mathrm{d}x}{x}$
- Countless generalizations ...

  higher dimensions, different step sizes, ...

Drunken birds



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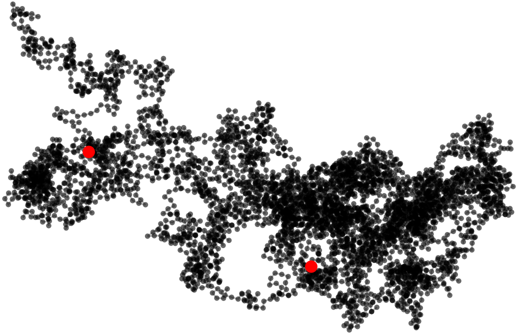

A drunk man will find his way home,
but a drunk bird may get lost forever.
Shizuo Kakutani, *1911–2004*

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

J. Borwein, D. Nuyens, A. Straub, J. Wan
Some arithmetic properties of short random walk integrals
The Ramanujan Journal, Vol. 26, Nr. 1, 2011, p. 109-132

J. Borwein, A. Straub, J. Wan
Three-step and four-step random walk integrals
Experimental Mathematics — to appear

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics — to appear

. . .

(Multiple) Mahler measure

DEF Kurokawa-Lalín-Ochiai

Multiple Mahler measure of polynomials $p_i(x_1, \ldots, x_n)$:

$$\mu(p_1, \ldots, p_k) := \int_{[0,1]^n} \prod_{i=1}^{k} \log \left| p_i \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}\mathbf{t}$$

$$\mu_k(p) := \int_{[0,1]^n} \log^k \left| p \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n} \right) \right| \mathrm{d}\mathbf{t}$$

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EG

$$W_n^{(k)}(0) = \mu_k(1 + x_1 + \ldots + x_{n-1})$$

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$$W_n^{(k)}(0) = \mu_k(1 + x_1 + \ldots + x_{n-1})$$

RK If the variables are independent, then

$$\mu(p_1, \ldots, p_n) = \mu(p_1) \cdots \mu(p_n).$$

- Representations for $W_n(s)$ give us, for instance,

$$\begin{aligned} W_n'(0) &= \log(2) - \gamma - \int_0^1 \left(J_0^n(x) - 1\right) \frac{\mathrm{d}x}{x} - \int_1^\infty J_0^n(x) \frac{\mathrm{d}x}{x} \\ &= \log(2) - \gamma - n \int_0^\infty \log(x) J_0^{n-1}(x) J_1(x) \mathrm{d}x. \end{aligned}$$

Moments of a 3-step random walk

EG Borwein–Borwein–S–Wan

$$\mu_1(1+x+y) = \frac{3}{2\pi}\operatorname{Ls}_2\left(\frac{2\pi}{3}\right)$$

$$\mu_2(1+x+y) = \frac{3}{\pi}\operatorname{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4}$$

$$\mu_3(1+x+y) \stackrel{?}{=} \frac{6}{\pi}\operatorname{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\operatorname{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi}{4}\operatorname{Cl}_2\left(\frac{\pi}{3}\right) - \frac{13}{2}\zeta(3)$$

$$\mu_4(1+x+y) \stackrel{?}{=} \frac{12}{\pi}\operatorname{Ls}_5\left(\frac{2\pi}{3}\right) - \frac{49}{3\pi}\operatorname{Ls}_5\left(\frac{\pi}{3}\right) + \frac{81}{\pi}\operatorname{Gl}_{4,1}\left(\frac{2\pi}{3}\right) + 3\pi\operatorname{Gl}_{2,1}\left(\frac{2\pi}{3}\right) + \frac{2}{\pi}\zeta(3)\operatorname{Cl}_2\left(\frac{\pi}{3}\right) + \operatorname{Cl}_2\left(\frac{\pi}{3}\right)^2 - \frac{29}{90}\pi^4$$

Derivatives of moments

- Using the residues $r_{5,k} = \operatorname{Res}_{-2k-2} W_5$:

$$p_5(x) = \sum_{k=0}^{\infty} r_{5,k}\, x^{2k+1}$$

EG

$$r_{5,0} = \frac{16 + 1140 W_5'(0) - 804 W_5'(2) + 64 W_5'(4)}{225},$$
$$r_{5,1} = \frac{26 r_{5,0} - 16 - 20 W_5'(0) + 4 W_5'(2)}{225}.$$

- Unfortunately, the Mahler measure $W_5'(0)$ “cancels” out.