Tools for special functions and special numbers

Graduate Student Colloquium
Tulane University, New Orleans

Armin Straub

October 15, 2013

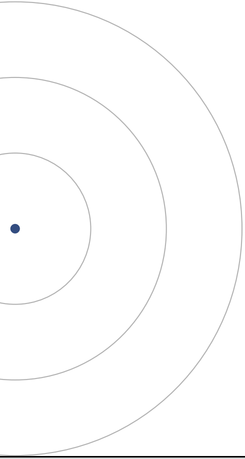
University of Illinois at Urbana–Champaign & Max-Planck-Institut für Mathematik, Bonn

ISC — PSLQ — OEIS — CAD — WZ

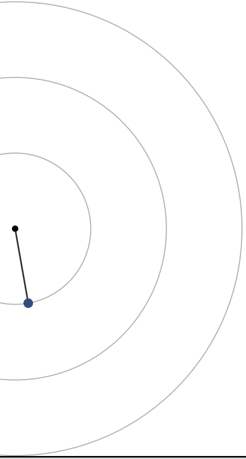
THE TOOLS TODAY

ISC	Inverse Symbolic Calculator
PSLQ	Lattice Reduction Algorithm
OEIS	On-Line Encyclopedia of Integer Sequences
CAD	Cylindrical Algebraic Decomposition
WZ	Wilf–Zeilberger Theory

Random walks

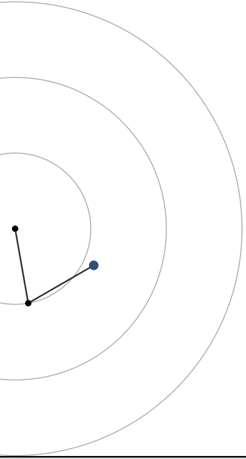


- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

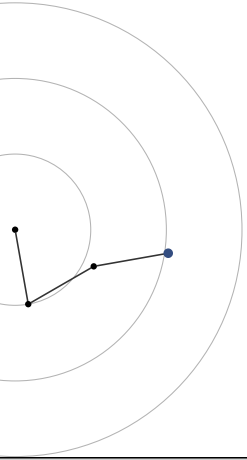


- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

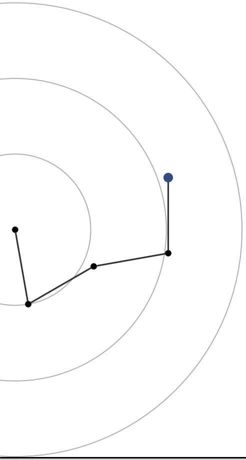
Random walks



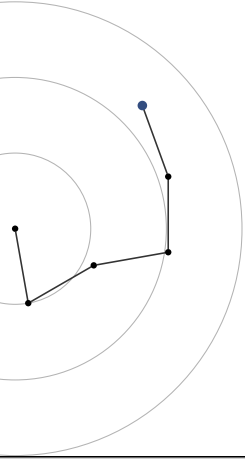
- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.



- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

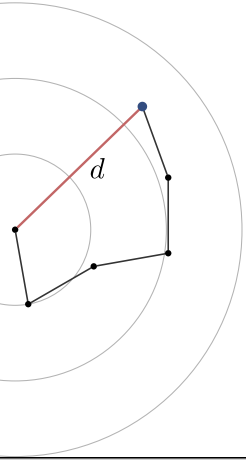


- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.



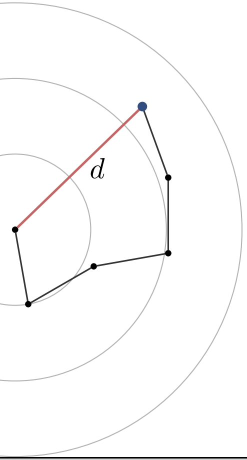
- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

Random walks



- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.

Random walks



- We study random walks in the plane consisting of n steps. Each step is of length 1 and is taken in a randomly chosen direction.
- We are interested in the distance traveled in n steps.

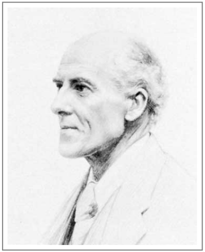
Q	For instance, how large is this distance on average?

- Probability density: $p_n(x)$

Random walks are only about 100 years old

- Karl Pearson asked for $p_n(x)$ in Nature in 1905.

  This famous question coined the term **random walk**.



The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

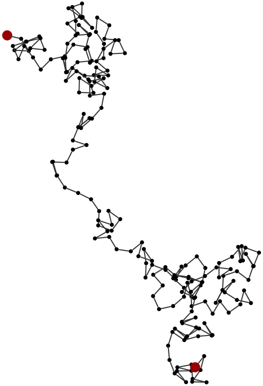
KARL PEARSON.

The Gables, East Ilsley, Berks.

Applications include:

- dispersion of mosquitoes
- random migration of micro-organisms
- phenomenon of laser speckle

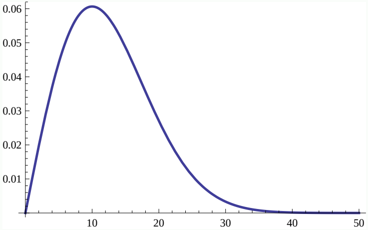
Long random walks



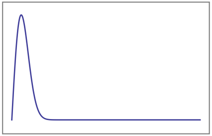
THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

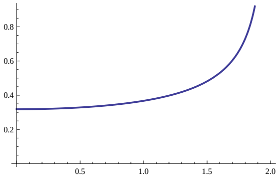
EG p_{200}



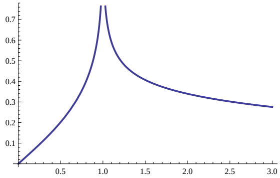
> *The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*
>
> **Karl Pearson**, *1905*



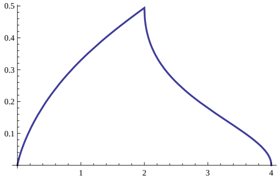
p_2



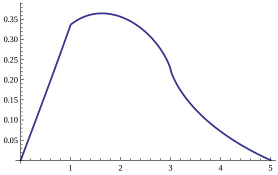
p_3



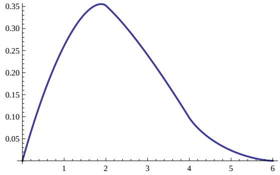
p_4



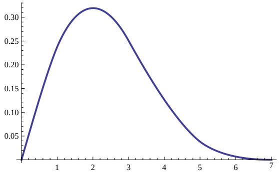
p_5



p_6

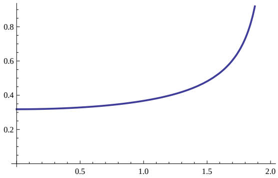


p_7

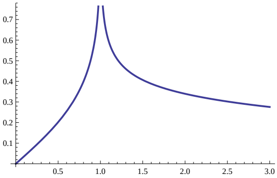


Densities of short walks

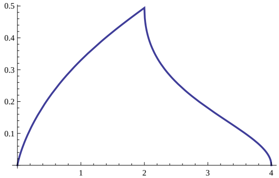
p_2



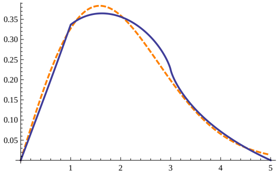
p_3



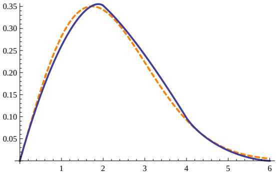
p_4



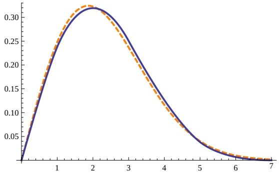
p_5



p_6



p_7



Moments

- The moments of a RV X are $E(X)$, $E(X^2)$, $E(X^3)$, $\ldots$
- If X has probability density $f(x)$ then

$$E(X^s) = \int_{-\infty}^{\infty} x^s f(x) \,\mathrm{d}x$$

FACT The moments $E(X^s)$ are analytic in s. (if, e.g., $f(x)$ is compactly supported)

Moments

- The moments of a RV X are $E(X)$, $E(X^2)$, $E(X^3)$, $\ldots$
- If X has probability density $f(x)$ then

$$E(X^s) = \int_{-\infty}^{\infty} x^s f(x)\,\mathrm{d}x$$

FACT The moments $E(X^s)$ are analytic in s. (if, e.g., $f(x)$ is compactly supported)

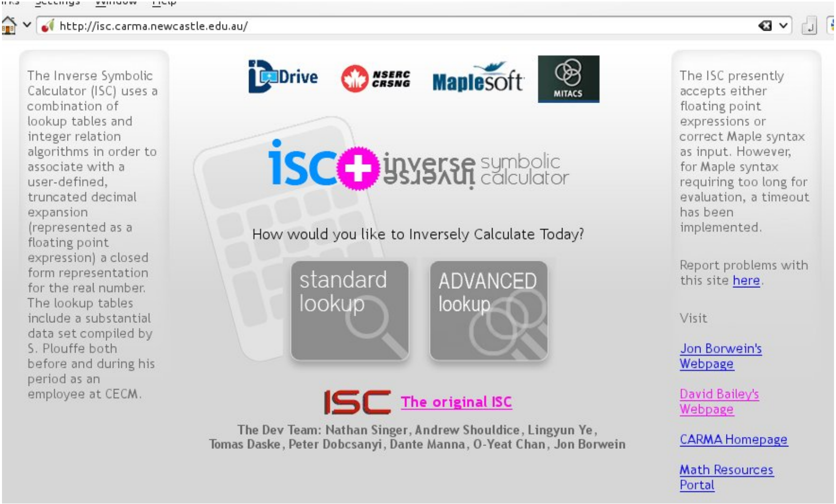
- Represent the kth step by the complex number $e^{2\pi i x_k}$.
- The sth moment of the distance after n steps is:

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

In particular, $W_n(1)$ is the average distance after n steps.

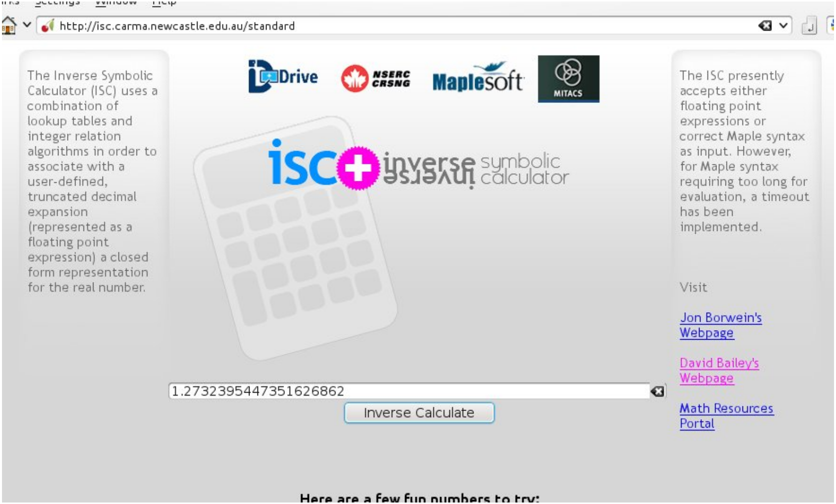
Average distance traveled in two steps

- Numerically: $W_2(1) \approx 1.27323954473351626862$



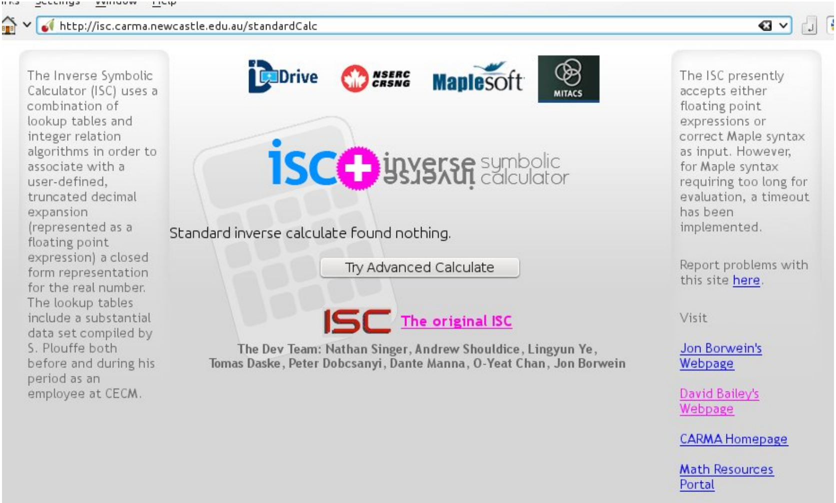
Average distance traveled in two steps

- Numerically: $W_2(1) \approx 1.2732395447351626862$



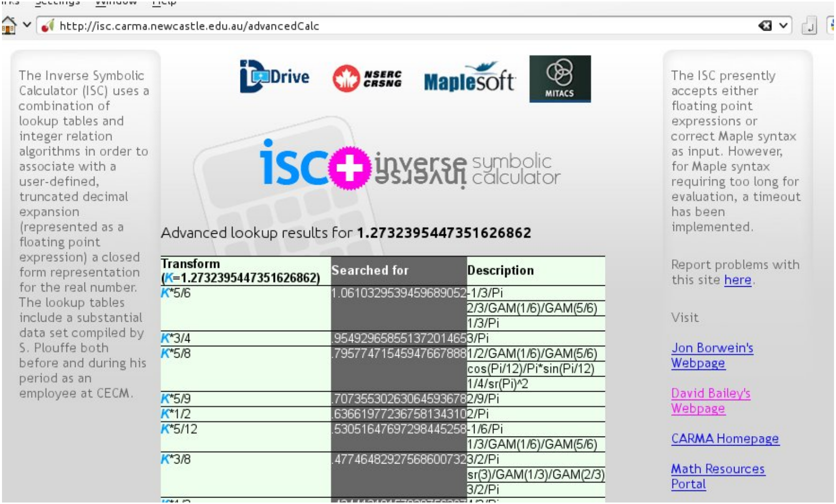
Average distance traveled in two steps

- Numerically: $W_2(1) \approx 1.27323954473516268 62$



Average distance traveled in two steps

- Numerically: $W_2(1) \approx 1.2732395447351626862$



Transform (K=1.2732395447351626862)	Searched for	Description
K*5/6	1.0610329539459689052	-1/3/Pi
		2/3/GAM(1/6)/GAM(5/6)
		1/3/Pi
K*3/4	.95492965855137201465	3/Pi
K*5/8	.79577471545947667888	1/2/GAM(1/6)/GAM(5/6)
		cos(Pi/12)/Pi*sin(Pi/12)
		1/4/sr(Pi)^2
K*5/9	.70735530263064593678	2/9/Pi
K*1/2	.63661977236758134310	2/Pi
K*5/12	.53051647697298445258	-1/6/Pi
		1/3/GAM(1/6)/GAM(5/6)
K*3/8	.47746482927568600732	3/2/Pi
		sr(3)/GAM(1/3)/GAM(2/3)
		3/2/Pi

The simple two-step case confirmed

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

The simple two-step case confirmed

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$
$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

The simple two-step case confirmed

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = \; ?$$

$$= \int_0^1 \left| 1 + e^{2\pi i y} \right| \mathrm{d}y$$

$$= \int_0^1 2\cos(\pi y)\mathrm{d}y$$

$$\left| 1 + e^{2\pi i y} \right|$$

$$= \left| 1 + (\cos \pi y + i \sin \pi y)^2 \right|$$

$$= 2\cos(\pi y)$$

The simple two-step case confirmed

- The average distance in two steps:

$$
\begin{aligned}
W_2(1) &= \int_0^1 \int_0^1 \left|e^{2\pi i x} + e^{2\pi i y}\right| \mathrm{d}x\mathrm{d}y = \ ? \\
&= \int_0^1 \left|1 + e^{2\pi i y}\right| \mathrm{d}y \\
&= \int_0^1 2\cos(\pi y)\mathrm{d}y \\
&= \frac{4}{\pi} \approx 1.27324
\end{aligned}
$$

$$
\begin{aligned}
&\left|1 + e^{2\pi i y}\right| \\
&= \left|1 + (\cos \pi y + i \sin \pi y)^2\right| \\
&= 2\cos(\pi y)
\end{aligned}
$$

The simple two-step case confirmed

- The average distance in two steps:

$$\left|1+e^{2\pi i y}\right| = \left|1+(\cos\pi y + i\sin\pi y)^2\right| = 2\cos(\pi y)$$

$$\begin{aligned} W_2(1) &= \int_0^1\int_0^1 \left|e^{2\pi i x}+e^{2\pi i y}\right| \mathrm{d}x\mathrm{d}y = \ ? \\ &= \int_0^1 \left|1+e^{2\pi i y}\right| \mathrm{d}y \\ &= \int_0^1 2\cos(\pi y)\mathrm{d}y \\ &= \frac{4}{\pi} \approx 1.27324 \end{aligned}$$

- Mathematica 7 and Maple 14 think the double integral is 0.

  Better: Mathematica 8 and 9 just don't evaluate the double integral.

The simple two-step case confirmed

- The average distance in two steps:

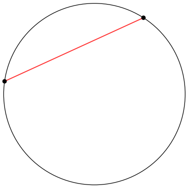
$$\left|1+e^{2\pi i y}\right| = \left|1+(\cos\pi y + i\sin\pi y)^2\right| = 2\cos(\pi y)$$

$$\begin{aligned} W_2(1) &= \int_0^1\int_0^1 \left|e^{2\pi i x}+e^{2\pi i y}\right| \mathrm{d}x\mathrm{d}y = \ ? \\ &= \int_0^1 \left|1+e^{2\pi i y}\right| \mathrm{d}y \\ &= \int_0^1 2\cos(\pi y)\mathrm{d}y \\ &= \frac{4}{\pi} \approx 1.27324 \end{aligned}$$

- Mathematica 7 and Maple 14 think the double integral is 0.

  Better: Mathematica 8 and 9 just don't evaluate the double integral.

- This is the average length of a random arc on a unit circle.



Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

- On a supercomputer: Lawrence Berkeley National Laboratory, 256 cores

$$W_5(1) \approx 2.0081618$$

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

- On a supercomputer: Lawrence Berkeley National Laboratory, 256 cores

$$W_5(1) \approx 2.0081618$$

- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

$W_3(1) = 1.57459723755189\ldots = ?$

Moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left| e^{2\pi i x_1} + \ldots + e^{2\pi i x_n} \right|^s \mathrm{d}\boldsymbol{x}$$

n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
6	2.194	6.000	18.91	66.00	248.8	996.0	4186.

$W_2(1) = \frac{4}{\pi}$

$W_3(1) = 1.57459723755189\ldots = ?$

For instance, the sequence $W_3(2k)$ is $1, 3, 15, 93, 639, 4653, \ldots$

The integer sequence database

This site is supported by donations to The OEIS Foundation.

Integer Sequences

1,3,15,93 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,3,15,93**

Displaying 1-8 of 8 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A002893 Sum_{k=0..n} binomial(n,k)^2 * binomial(2k,k). +20 21

(Formerly M2998 N1214)

1, **3**, **15**, **93**, 639, 4653, 35169, 272835, 2157759, 17319837, 140668065, 1153462995, 9533639025, 79326566595, 663835030335, 5582724468093, 47152425626559, 399769750195965, 3400775573443089, 29016970072920387, 248256043372999089 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS
```
Comment from Matthijs Coster, Apr 28 2004: This is the Taylor expansion of a special
  point on a curve described by Beauville.
a(n) is the (2n)th moment of the distance from the origin of a 3-step random walk in
  the plane - Peter M. W. Gill (peter.gill(AT)nott.ac.uk), Feb 27 2004
a(n) is the number of Abelian squares of length 2n over a 3-letter alphabet. [From
  _Jeffrey Shallit_, Aug 17 2010]
Consider 2D simple random walk on honeycomb lattice. a(n) gives number of paths of
  length 2n ending at origin - Sergey Perepechko Feb 16 2011
Row sums of the square of A008459. - Peter Bala, Mar 05 2013
Conjecture: For each n=1,2,3,... the polynomial g_n(x) = sum_{k=0}^n
  binomial(n,k)^2*binomial(2k,k)*x^k is irreducible over the field of rational
  numbers. [Zhi-Wei Sun, Mar 21 2013]
```

REFERENCES
```
David H. Bailey, Jonathan M. Borwein, David Broadhurst and M. L. Glasser, Elliptic
  integral evaluations of Bessel moments, arXiv:0801.0891.
P. Barrucand, A combinatorial identity, Problem 75-4, SIAM Rev., 17 (1975), 168.
```

The integer sequence database

This site is supported by donations to The OEIS Foundation.

Integer Sequences

1,5,45,545 [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:1,5,45,545**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A169714 The function W_5(2n) (see Borwein et al. reference for definition). +20 5

1, 5, 45, 545, 7885, 127905, 2241225, 41467725, 798562125, 15855173825, 322466645545, 6687295253325, 140927922498025, 3010302779775725, 65046639827565525, 1419565970145097545, 31249959913055650125, 693192670456484513025 (list; graph; refs; listen; history; text; internal format)

OFFSET	0,2
COMMENTS	Row sums of the fourth power of A008459. - Peter Bala, Mar 05 2013
REFERENCES	Armin Straub, Arithmetic aspects of random walks and methods in definite integration, Ph. D. Dissertation, School Of Science And Engineering, Tulane University, 2012. - From N. J. A. Sloane, Dec 16 2012
LINKS	Table of n, a(n) for n=0..17. Jonathan M. Borwein, Dirk Nuyens, Armin Straub and James Wan, Random Walk Integrals, 2010. Jonathan M. Borwein and Armin Straub, Mahler measures, short walks and log-sine integrals (2012)
FORMULA	Sum_{n>=0} a(n)*x^n/n!^2 = (Sum_{n>=0} x^n/n!^2)^5 = BesselI(0, 2*sqrt(x))^5. - Peter Bala, Mar 05 2013
MAPLE	A169714 := proc(n) W(5, 2*n) ; end proc: # with W() from A169715, R. J. Mathar, Mar 27 2012
CROSSREFS	Cf. A002893, A002895, A169715.

Advertisement

- Based on the observation that

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

knowledge of **modular forms** allows us to deduce:

THM Borwein-Nuyens-S-Wan, 2010

$$W_3(1) = \frac{3}{16} \frac{2^{1/3}}{\pi^4} \Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4} \frac{2^{2/3}}{\pi^4} \Gamma^6\left(\frac{2}{3}\right)$$
$$= 1.57459723755189\ldots$$

Modular forms

> *Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents. But they do exist.*
>
> **Barry Mazur** (BBC Interview, "The Proof", 1997)

DEF Actions of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

- on $\tau \in \mathcal{H}$ by $\gamma \cdot \tau = \dfrac{a\tau + b}{c\tau + d}$,
- on $f : \mathcal{H} \to \mathbb{C}$ by $(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma \cdot \tau)$.

EG $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$T \cdot \tau = \tau + 1, \qquad S \cdot \tau = -\frac{1}{\tau}$$

Modular forms

> *There's a saying attributed to Eichler that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.*
>
> **Andrew Wiles** (BBC Interview, "The Proof", 1997)

DEF A function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of weight k if

- $f|_k\gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,
- f is holomorphic (including at the cusp $i\infty$).

EG

$$f(\tau + 1) = f(\tau), \qquad \tau^{-k} f(-1/\tau) = f(\tau).$$

- Similarly, MFs w.r.t. finite-index $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$
- Spaces of MFs finite dimensional, Hecke operators, ...

- The **Dedekind eta function** $(q = e^{2\pi i\tau})$

$$\eta(\tau) = q^{1/24} \prod_{n\geqslant 1}(1 - q^n)$$

transforms as

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau).$$

EG $\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24}$ is a modular form of weight 12.

Modularity of the three-step moments

- The even moments $1, 3, 15, 93, 639, \ldots$

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

have the **modular parametrization**

$$\underbrace{\frac{\eta^6(2\tau)\eta(3\tau)}{\eta^3(\tau)\eta^2(6\tau)}}_{\text{modular form}} = \sum_{k \geqslant 0} W_3(2k) \underbrace{\left(\frac{\eta(\tau)\eta^2(6\tau)}{\eta^2(2\tau)\eta(3\tau)}\right)}_{\text{modular function}}^{4k}.$$

Modularity of the three-step moments

- The even moments $1, 3, 15, 93, 639, \ldots$

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

have the **modular parametrization**

$$\underbrace{\frac{\eta^6(2\tau)\eta(3\tau)}{\eta^3(\tau)\eta^2(6\tau)}}_{\text{modular form}} = \sum_{k\geqslant 0} W_3(2k) \underbrace{\left(\frac{\eta(\tau)\eta^2(6\tau)}{\eta^2(2\tau)\eta(3\tau)}\right)}_{\text{modular function}}^{4k}.$$

EG The values of modular functions at quadratic irrationalities $\tau \in \mathbb{Q}(\sqrt{-d})$ are algebraic!

PSLQ predicts that for the above modular function $x(\tau)$, the value $x(i/3) \approx 0.52754$ has minimal polynomial $1 - 6x^4 - 24x^6 - 3x^8$.

Integer relation algorithms

- How does the ISC recognize numbers?

Integer relation algorithms

- How does the ISC recognize numbers?
- **PSLQ** takes numbers $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and tries to find integers $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$, not all zero, such that

$$\boldsymbol{x} \cdot \boldsymbol{m} = m_1 x_1 + \ldots + m_n x_n = 0.$$

The vector $\boldsymbol{m}$ is called an **integer relation** for $\boldsymbol{x}$.

In case that no relation is found, PSLQ provides a lower bound for the norm of any potential integer relation.

- How does the ISC recognize numbers?
- **PSLQ** takes numbers $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and tries to find integers $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$, not all zero, such that

$$\boldsymbol{x} \cdot \boldsymbol{m} = m_1 x_1 + \ldots + m_n x_n = 0.$$

The vector $\boldsymbol{m}$ is called an **integer relation** for $\boldsymbol{x}$.

In case that no relation is found, PSLQ provides a lower bound for the norm of any potential integer relation.

EG Is $x = 0.31783724519578224473\ldots$ algebraic?

In[1]:= $\mathbf{PSLQ[\{1, \boldsymbol{x}, \boldsymbol{x}^2, \boldsymbol{x}^3, \boldsymbol{x}^4\}]}$

Out[1]= $\{1, 0, -10, 0, 1\}$

That is, x likely has minimal polynomial $1 - 10x^2 + x^4$.

Therefore, $x = \sqrt{3} - \sqrt{2}$.

Using PSLQ to find functional relations

- A well-known fact: $\sin((2n-1)x)$ is a linear combination of $\sin(x)$, $\sin^3(x)$, …, $\sin^{2n-1}(x)$

Using PSLQ to find functional relations

- A well-known fact: $\sin((2n-1)x)$ is a linear combination of $\sin(x)$, $\sin^3(x)$, ..., $\sin^{2n-1}(x)$

EG

In[1]:= $\mathbf{With[\{x = 1\}, PSLQ[}$
$\mathbf{N[\{Sin[5x], Sin[x], Sin[x]^3, Sin[x]^5\}, 20]]]}$

Out[1]= $\{-1, 5, -20, 16\}$

In other words,

$$\sin(5x) = 5\sin(x) - 20\sin^3(x) + 16\sin^5(x).$$

Cylindrical Algebraic Decomposition

EG Arithmetic mean $\geqslant$ geometric mean

In[1]:= $\mathbf{CylindricalDecomposition}[(\boldsymbol{a}+\boldsymbol{b})/\mathbf{2} \geqslant \mathbf{Sqrt}[\boldsymbol{ab}], \{\boldsymbol{a}, \boldsymbol{b}\}]$

Out[1]= $a \geqslant 0 \wedge b \geqslant 0$

EG Arithmetic mean $\geqslant$ geometric mean

In[1]:= $\mathbf{CylindricalDecomposition}[(a+b)/2 \geqslant \mathbf{Sqrt}[ab], \{a, b\}]$

Out[1]= $a \geqslant 0 \wedge b \geqslant 0$

EG If the sum of four positive numbers is $4c$ and the sum of their squares is $8c^2$, then none of the numbers can exceed $(1+\sqrt{3})c$.

EG Arithmetic mean $\geqslant$ geometric mean

In[1]:= $\mathbf{CylindricalDecomposition[(a+b)/2 \geqslant Sqrt[ab], \{a, b\}]}$

Out[1]= $a \geqslant 0 \wedge b \geqslant 0$

EG If the sum of four positive numbers is $4c$ and the sum of their squares is $8c^2$, then none of the numbers can exceed $(1+\sqrt{3})c$.

In[2]:= $\mathbf{CylindricalDecomposition[Exists[\{a_2, a_3, a_4\},}$
$\mathbf{a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 > 0 \wedge}$
$\mathbf{a_1 + a_2 + a_3 + a_4 == 4c \wedge}$
$\mathbf{a_1^2 + a_2^2 + a_3^2 + a_4^2 == 8c^2], \{c, a_1\}]}$

Out[2]= $c > 0 \wedge 2c < a_1 \leqslant (1+\sqrt{3})c$

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG An obviously positive rational function:

$$\frac{1}{1 - x - y + xy} = \frac{1}{(1-x)(1-y)}$$

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG An obviously positive rational function:

$$\frac{1}{1 - x - y + xy} = \frac{1}{(1 - x)(1 - y)}$$

THM

$$\frac{1}{1 - x - y + \lambda xy}$$

is positive if and only if $\lambda \leqslant 1$.

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG An obviously positive rational function:

$$\frac{1}{1 - x - y + xy} = \frac{1}{(1-x)(1-y)}$$

CONJ Askey–Gasper 1972 The following rational function is positive:

$$\frac{1}{1 - (x + y + z + w) + \frac{2}{3}(xy + xz + xw + yz + yw + zw)}$$

This is a rescaled version of $1/e_2(1-x, 1-y, 1-z, 1-w)$.

Positivity of rational functions

EG The Askey–Gasper rational function $A(x,y,z)$ and the Szegő rational function $S(x,y,z)$ are positive.

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

THM S 2007 There is a **positivity-preserving** operator T such that $T \cdot A = S$.

Positivity of rational functions

EG The Askey–Gasper rational function $A(x,y,z)$ and the Szegő rational function $S(x,y,z)$ are positive.

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

THM S 2007 There is a **positivity-preserving** operator T such that $T \cdot A = S$.

EG The diagonal Taylor terms of A are given by

$$[x^n y^n z^n] A(x,y,z) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

By **WZ**, both sides satisfy the recurrence

$$(n+1)^2 a_{n+1} = (7n^2+7n+2)a_n + 8n^2 a_{n-1}.$$

Positivity of rational functions

EG The diagonal Taylor terms of $S(2x,2y,2z)$, namely

$$1, 12, 198, 3720, 75690, 1626912, \ldots,$$

satisfy the recurrence

$$2(n+1)^2 s_{n+1} = 3\left(27n^2+27n+8\right) s_n - 81(3n-1)(3n+1)s_{n-1}.$$

EG The diagonal Taylor terms of $S(2x, 2y, 2z)$, namely

$$1, 12, 198, 3720, 75690, 1626912, \ldots,$$

satisfy the recurrence

$$2(n+1)^2 s_{n+1} = 3\left(27n^2 + 27n + 8\right) s_n - 81(3n-1)(3n+1)s_{n-1}.$$

To prove positivity from the recurrence, apply CAD to the formula

$$(\forall n, A, B) \quad n \geqslant 1, A \geqslant 0, B \geqslant \lambda A \quad \Longrightarrow \quad C \geqslant \lambda B$$

where $2(n+1)^2 C = 3(27n^2 + 27n + 8)B - 81(3n-1)(3n+1)A.$

Positivity of rational functions

EG The diagonal Taylor terms of $S(2x, 2y, 2z)$, namely

$$1, 12, 198, 3720, 75690, 1626912, \ldots,$$

satisfy the recurrence

$$2(n+1)^2 s_{n+1} = 3\left(27n^2 + 27n + 8\right) s_n - 81(3n-1)(3n+1)s_{n-1}.$$

To prove positivity from the recurrence, apply CAD to the formula

$$(\forall n, A, B) \quad n \geqslant 1, A \geqslant 0, B \geqslant \lambda A \quad \Longrightarrow \quad C \geqslant \lambda B$$

where $2(n+1)^2 C = 3(27n^2 + 27n + 8)B - 81(3n-1)(3n+1)A.$

In[1]:= $\mathbf{With[\{C = \ldots\},}$
$\mathbf{CylindricalDecomposition[ForAll[\{n, A, B\},}$
$\mathbf{n \geqslant 1 \wedge B \geqslant \lambda A \wedge A \geqslant 0, C \geqslant \lambda B], \{\lambda\}]]}$

Out[1]= $27/2 \leqslant \lambda \leqslant 3/8(31 + \sqrt{385})$

Positivity of rational functions

- The Kauers–Zeilberger rational function

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

  is conjectured to be positive.
- Its positivity implies the positivity of the Askey–Gasper function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

PROP S-Zudilin 2013 The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Positivity of rational functions

Q Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)}$$

implied by the positivity of its diagonal?

- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

Positivity of rational functions

Q Under what condition(s) is the positivity of a rational function

$$h(x_1, \ldots, x_d) = \frac{1}{\sum_{k=0}^{d} c_k e_k(x_1, \ldots, x_d)}$$

implied by the positivity of its diagonal?

- Is the positivity of $h(x_1, \ldots, x_{d-1}, 0)$ a sufficient condition?

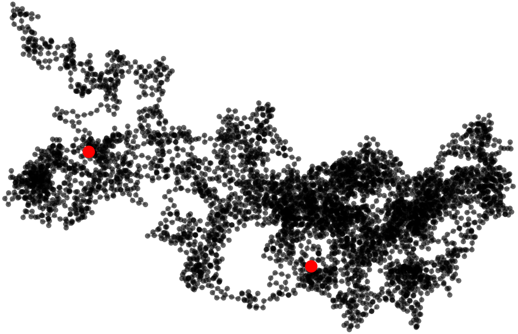
EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

THM S-Zudilin 2013

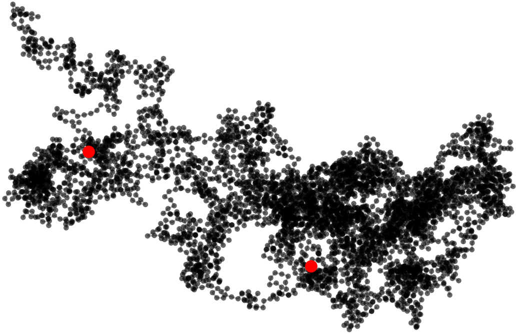
$$h(x, y) = \frac{1}{1 + c_1(x + y) + c_2 xy}$$

is positive iff $h(x, 0)$ and the diagonal of $h(x, y)$ are positive.

Drunken birds



Drunken birds



> *A drunk man will find his way home,*
> *but a drunk bird may get lost forever.*
> **Shizuo Kakutani**, *1911–2004*

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
Preprint, 2013

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

J. Borwein, D. Nuyens, A. Straub, J. Wan
Some arithmetic properties of short random walk integrals
The Ramanujan Journal, Vol. 26, Nr. 1, 2011, p. 109-132

A. Straub
Positivity of Szegö's rational function
Advances in Applied Mathematics, Vol. 41, Issue 2, Aug 2008, p. 255-264