On a secant Dirichlet series and Eichler integrals of Eisenstein series

Modular Forms and Modular Integrals in Memory of Marvin Knopp
AMS Sectional Meeting, Temple University

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Based on joint work with:

Bruce Berndt
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- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

EG **LRR '13**

$$\psi_2(\sqrt{2}) = -\frac{\pi^2}{3}, \qquad \psi_2(\sqrt{6}) = \frac{2\pi^2}{3}$$

Secant zeta function

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CONJ
LRR '13

For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

Secant zeta function: Motivation

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

- Half of the Clausen and Glaisher functions reduce, e.g.,

$$\sum_{n=1}^{\infty} \frac{\cos(n\tau)}{n^{2m}} = \operatorname{poly}_m(\tau), \qquad \operatorname{poly}_1(\tau) = \frac{\tau^2}{4} - \frac{\pi\tau}{2} + \frac{\pi^2}{6}.$$

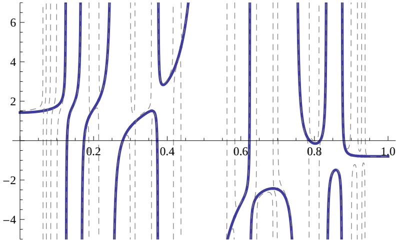
- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

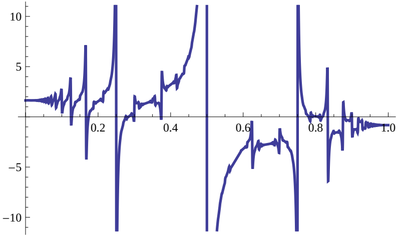
In fact, this was already included in a general formula by Lerch.

Secant zeta function: Convergence

- $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ has singularity at rationals with even denominator



$\psi_2(\tau)$ truncated to 4 and 8 terms



Re $\psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/1000$

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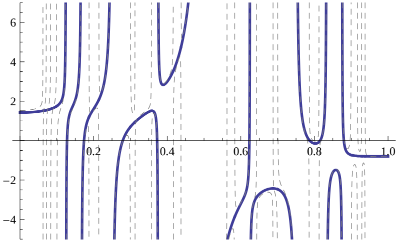

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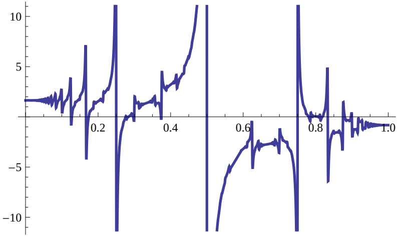

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THM Lalín–Rodrigue–Rogers 2013

The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Secant zeta function: Functional equation

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau + 2) = \psi_s(\tau)$.

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$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
$$= \pi^{2m}\,\mathrm{rat}(\tau)$$

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THM LRR, BS 2013

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proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\mathrm{radius}(C) \to \infty$. □

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DEF slash operator

$$F|_k\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) = (c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right)$$

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THM LRR, BS 2013

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- In terms of

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the functional equations become

$$\psi_{2m}|_{1-2m}(T^2-1) = 0,$$
$$\psi_{2m}|_{1-2m}(R^2-1) = \pi^{2m}\operatorname{rat}(\tau).$$

Secant zeta function: Functional equation

- The matrices

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad R^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

together with $-I$, generate

$$\Gamma(2) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \quad \gamma \equiv I \pmod 2\}.$$

COR For any $\gamma \in \Gamma(2)$,

$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m}\,\mathrm{rat}(\tau).$$

THM LRR, BS 2013

For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

proof

- Note that
$$\begin{pmatrix} X & rY \\ Y & X \end{pmatrix} \cdot \sqrt{r} = \sqrt{r}.$$
- As shown by Lagrange, there are X and Y which solve Pell's equation
$$X^2 - rY^2 = 1.$$

$\square$

Secant zeta function: Special values

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For positive integers m, r,

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- Note that
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- As shown by Lagrange, there are X and Y which solve Pell's equation
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- Since
$$\gamma = \begin{pmatrix} X & rY \\ Y & X \end{pmatrix}^2 = \begin{pmatrix} X^2 + rY^2 & 2rXY \\ 2XY & X^2 + rY^2 \end{pmatrix} \in \Gamma(2),$$
the claim follows from the evenness of ψ_{2m} and
$$\psi_{2m}|_{1-2m}(\gamma - 1) = \pi^{2m} \operatorname{rat}(\tau).$$
□

Eichler integrals

- F is an **Eichler integral** if $D^{k-1}F$ is modular of weight k.
- Such Eichler integrals are characterized by

$$F|_{2-k}(\gamma - 1) = \mathrm{poly}(\tau), \qquad \deg \mathrm{poly} \leqslant k - 2.$$

- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.

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- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.
- For a modular form $f(\tau) = \sum a(n) q^n$ of weight k, define

$$\begin{aligned}\tilde{f}(\tau) &= \int_{\tau}^{i\infty} [f(z) - a_0] (z-\tau)^{k-2} \mathrm{d}z \\ &= \frac{(-1)^k \Gamma(k-1)}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n.\end{aligned}$$

If $a_0 = 0$, $\tilde{f}$ is an Eichler integral as defined above.

Eichler integrals of Eisenstein series

- For the **Eisenstein series** G_{2k},

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{\Gamma(2k)} \underbrace{\sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n}_{\sum \frac{n^{2k-1}q^n}{1-q^n}},$$

$$\tilde{G}_{2k}(\tau) = \frac{4\pi i}{2k-1} \underbrace{\sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^{2k-1}} q^n}_{\sum \frac{n^{1-2k}q^n}{1-q^n}}.$$

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- The period "polynomial" $\tilde{G}_{2k}|_{2-2k}(S-1)$ is given by

$$\frac{(2\pi i)^{2k}}{2k-1} \left[\sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} X^{2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}} (X^{2k-2} - 1) \right].$$

Ramanujan's formula

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\alpha n}-1}\right\} = (-\beta)^{-m}\left\{\frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty}\frac{n^{-2m-1}}{e^{2\beta n}-1}\right\}$$

$$-2^{2m}\sum_{n=0}^{m+1}(-1)^n\frac{B_{2n}}{(2n)!}\frac{B_{2m-2n+2}}{(2m-2n+2)!}\alpha^{m-n+1}\beta^n.$$

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$$-2^{2m} \sum_{n=0}^{m+1} (-1)^n \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} \alpha^{m-n+1} \beta^n.$$

$$\frac{1}{e^x - 1} = \frac{1}{2} \cot(\tfrac{x}{2}) - \frac{1}{2}$$

- In terms of

$$\xi_s(\tau) = \sum_{n=1}^{\infty} \frac{\cot(\pi n \tau)}{n^s},$$

Ramanujan's formula takes the form

$$\xi_{2k-1}|_{2-2k}(S-1) = (-1)^k (2\pi)^{2k-1} \sum_{s=0}^{k} \frac{B_{2s}}{(2s)!} \frac{B_{2k-2s}}{(2k-2s)!} \tau^{2s-1}.$$

Secant zeta function

- $\sum \frac{\cot(\pi n \tau)}{n^{2k-1}}$ is an Eichler integral of the Eisenstein series G_{2k}.

EG

$$\cot(\pi\tau) = \frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{1}{\tau + j}$$

$$\lim_{N \to \infty} \sum_{j=-N}^{N}$$

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$$\lim_{N \to \infty} \sum_{j=-N}^{N}$$

- $\sum \frac{\sec(\pi n \tau)}{n^{2k}}$ is an Eichler integral of an Eisenstein series with character.

EG

$$\sec\left(\frac{\pi\tau}{2}\right) = \frac{2}{\pi} \sum_{j \in \mathbb{Z}} \frac{\chi_{-4}(j)}{\tau + j}$$

- $\displaystyle\sum_{m,n \in \mathbb{Z}}{}' \frac{\chi_{-4}(n)}{(m\tau + n)^{2k+1}}$ is an Eisenstein series of weight $2k + 1$.

Eisenstein series

- More generally, we have the Eisenstein series

$$E_k(\tau;\chi,\psi) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\chi(m)\psi(n)}{(m\tau+n)^k},$$

  where χ and ψ are Dirichlet characters modulo L and M.
- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

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PROP Modular transformations: $\gamma = \begin{pmatrix} a & Mb \\ Lc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

- $E_k(\tau;\chi,\psi)|_k\gamma = \chi(d)\bar{\psi}(d)E_k(\tau;\chi,\psi)$
- $E_k(\tau;\chi,\psi)|_kS = \chi(-1)E_k(\tau;\psi,\chi)$

PROP If ψ is primitive, the L-function of $E(\tau) = E_k(\tau;\chi,\psi)$ is

$$L(E,s) = \mathrm{const}\cdot M^s L(\chi,s)L(\bar{\psi},1-k+s).$$

Generalized Bernoulli numbers

EG
Euler

$$\zeta(2n) = -\frac{1}{2}(2\pi i)^{2n} \frac{B_{2n}}{(2n)!}$$

- For integer $n > 0$ and primitive χ with $\chi(-1) = (-1)^n$,
  (χ of conductor L and Gauss sum $G(\chi)$)

$$L(n, \chi) = (-1)^{n-1} \frac{G(\chi)}{2} \left(\frac{2\pi i}{L}\right)^n \frac{B_{n,\bar{\chi}}}{n!},$$

$$L(1-n, \chi) = -B_{n,\chi}/n.$$

- The **generalized Bernoulli numbers** have generating function

$$\sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!} = \sum_{a=1}^{L} \frac{\chi(a) x e^{ax}}{e^{Lx} - 1}.$$

Period polynomials of Eisenstein series

THM Berndt-S 2013

For $k \geqslant 3$ and primitive $\chi \neq 1$, $\psi \neq 1$,

$$\tilde{E}_k(X;\chi,\psi) - \psi(-1) X^{k-2} \tilde{E}_k(-1/X;\psi,\chi) = \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)! L^{k-s}} \frac{B_{s,\bar{\psi}}}{s! M^s} X^{s-1}.$$

$$\text{const} = -\chi(-1) G(\chi) G(\psi) \frac{(2\pi i)^k}{k-1}$$

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COR Berndt-S 2013

For $k \geqslant 3$, primitive χ, $\psi \neq 1$, and n such that $L|n$,

$$R = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$$

$$\tilde{E}_k(X;\chi,\psi)|_{2-k}(1-R^n) = \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}(1-(nX+1)^{k-1-s}).$$

Unimodular polynomials

DEF $p(x)$ is **unimodular** if all its zeros have absolute value 1.

- Kronecker: if $p(x) \in \mathbb{Z}[x]$ is monic and unimodular, then all nonzero roots are roots of unity.

EG

$$x^2 + \frac{6}{5}x + 1 = \left(x + \tfrac{3+4i}{5}\right)\left(x + \tfrac{3-4i}{5}\right)$$

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THM Cohn 1922 $P(x)$ is unimodular if and only if

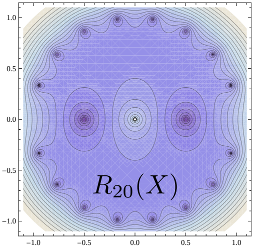
- $P(x) = a_0 + a_1 x + \ldots + a_n x^n$ is self-inversive, i.e. $a_k = \varepsilon \overline{a_{n-k}}$ for some $|\varepsilon| = 1$, and
- $P'(x)$ has all its roots within the unit circle.

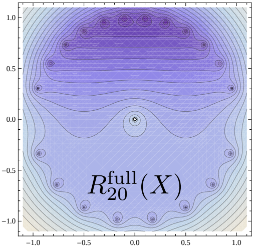
- Following Gun–Murty–Rath, the **Ramanujan polynomials** are

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

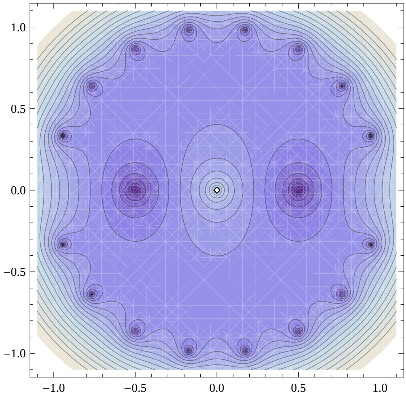
THM Murty-Smyth-Wang '11: All nonreal zeros of $R_k(X)$ lie on the unit circle.
For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.

THM Lalín-Smyth '13: $R_{2k}(X) + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.

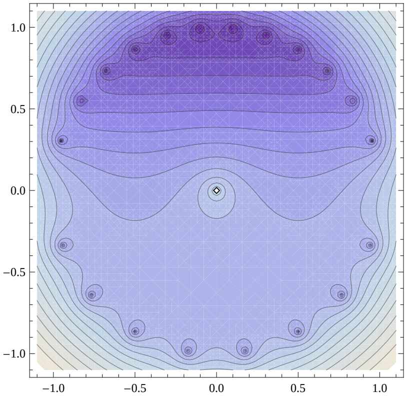




Ramanujan polynomials

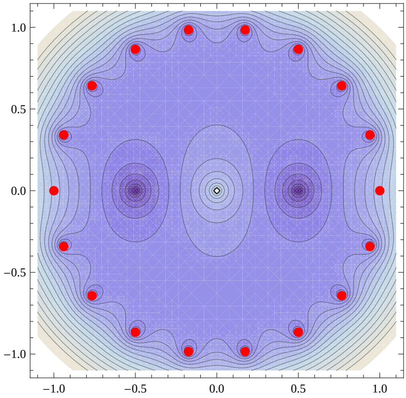


$R_{20}(X)$



$R_{20}^{\text{full}}(X)$

Ramanujan polynomials



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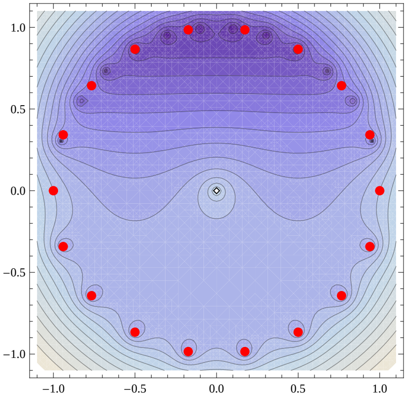

$R_{20}^{\text{full}}(X)$

Generalized Ramanujan polynomials

- We consider two kinds of **generalized Ramanujan polynomials**:

$$S_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{LX}{M}\right)^{k-s-1}$$

$$R_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M}\right)^{k-s-1} (1 - X^{s-1})$$

- Obviously, $S_k(X;1,1) = R_k(X)$.

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- Obviously, $S_k(X;1,1) = R_k(X)$.

PROP Berndt-S 2013

- For $k > 1$, $R_{2k}(X;1,1) = R_{2k}(X)$.
- $R_k(X;\chi,\psi)$ is self-inversive.

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- Obviously, $S_k(X;1,1) = R_k(X)$.

PROP Berndt-S 2013

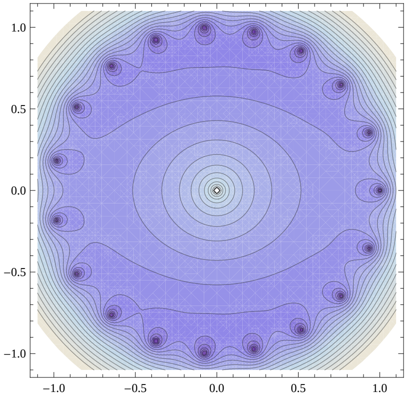
- For $k > 1$, $R_{2k}(X;1,1) = R_{2k}(X)$.
- $R_k(X;\chi,\psi)$ is self-inversive.

CONJ Berndt-S 2013

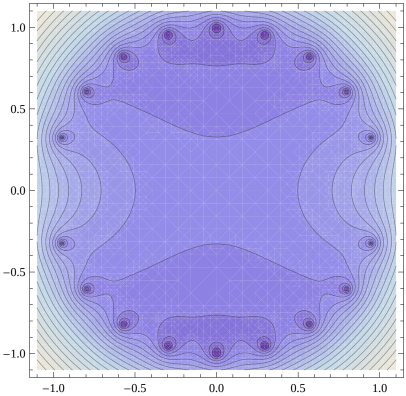
Let χ, ψ be nonprincipal real Dirichlet characters.

- $R_k(X;\chi,\psi)$ is unimodular.
- $S_k(X;\chi,\chi)$ is unimodular (up to trivial zero roots).

Generalized Ramanujan polynomials

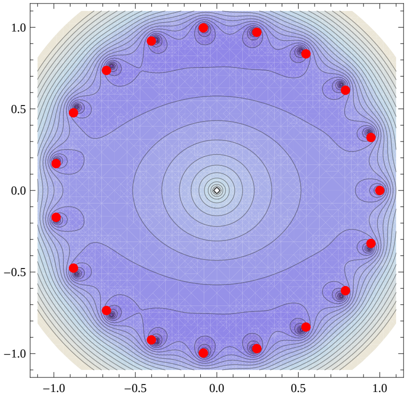


$R_{19}(X; 1, \chi_{-4})$

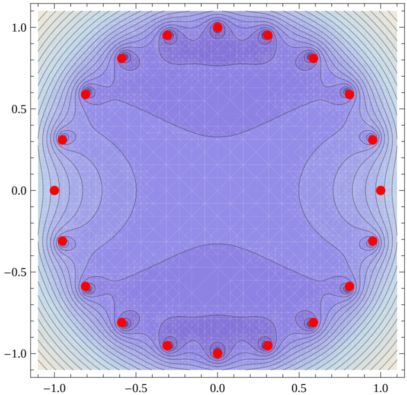


$S_{20}(X; \chi_{-4}, \chi_{-4})$

Generalized Ramanujan polynomials



$R_{19}(X; 1, \chi_{-4})$



$S_{20}(X; \chi_{-4}, \chi_{-4})$

Unimodularity of period polynomials

- Both kinds of generalized Ramanujan polynomials are, essentially, period polynomials: χ, ψ primitive, nonprincipal

$$\begin{aligned} S_k(X;\chi,\psi) &= \text{const} \cdot \left[\tilde{E}_k(X;\bar{\chi},\bar{\psi}) - \psi(-1) X^{k-2} \tilde{E}_k(-1/X;\bar{\psi},\bar{\chi}) \right] \\ R_k(LX+1;\chi,\psi) &= S_k(X;\chi,\psi)|_{2-k}(1-R^L) \\ &= \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L) \end{aligned}$$

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$$= \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L)$$

THM Conrey-Farmer-Imamoglu 2012

For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm\frac{1}{2}$,
- and all remaining zeros lie on the unit circle.

THM El-Guindy–Raji 2013

For any Hecke eigenform (for $\mathrm{SL}_2(\mathbb{Z})$), the full period polynomial has all zeros on the unit circle.

Application: Grosswald-type formula for Dirichlet L-values

THM Berndt-S 2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$,
($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

$$L(k-1, \chi) = \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k \left(\frac{\alpha - 1}{L}; \chi, 1 \right) - \alpha^{k-2} \tilde{E}_k \left(\frac{1 - 1/\alpha}{L}; \chi, 1 \right) \right]$$

$$= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right].$$

THM Berndt-S 2013

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THM Gun–Murty–Rath 2011

As $\beta \in \mathcal{H}$, $\beta^{2k-2} \neq 1$, ranges over algebraic numbers, the values

$$\frac{1}{\pi} \left[\tilde{E}_{2k}(\beta; 1, 1) - \beta^{2k-2} \tilde{E}_{2k}(-1/\beta; 1, 1) \right]$$

contain at most one algebraic number.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2013