A solution of Sun's $520 challenge concerning $\frac{520}{\pi}$

27th Automorphic Forms Workshop, Dublin

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March 14, 2013

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Based on joint work with:

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CONJ

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- roughly, each two terms of the outer sum give one correct digit

> *I would like to offer $520 (520 US dollars) for the person who could give the first correct proof of (*) in 2012 because May 20 is the day for Nanjing University.*
> **Zhi-Wei Sun** (2011)

The earliest series for $1/\pi$?

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(-1)^n(4n+1)$$

- Included in first letter of Ramanujan to Hardy

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  but already given by Bauer in 1859 and further studied by Glaisher
- Limiting case of the terminating (Zeilberger, 1994)

$$\frac{\Gamma(3/2+m)}{\Gamma(3/2)\Gamma(m+1)} = \sum_{n=0}^{\infty} \frac{(1/2)_n^2(-m)_n}{n!^2(3/2+m)_n}(-1)^n(4n+1)$$

which has a WZ proof

Carlson's theorem justifies setting $m = -1/2$.

The first of Ramanujan's series

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

- Starred in High School Musical, a 2006 Disney production
- Both series also have WZ proof (Guillera, 2006)

  but no such proof known for more general series

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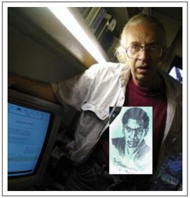
Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$



- Instead of proof, Ramanujan hints at "corresponding theories" which he unfortunately never developed
- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

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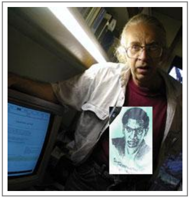
- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.

- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers



Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

The Chudnovsky series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)! n!^3} \frac{13591409 + 545140134n}{640320^{3n+3/2}}$$

- Used by David and Gregory Chudnovsky in 1988 to compute $2,260,331,336$ digits of π

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- Used by David and Gregory Chudnovsky in 1988 to compute $2,260,331,336$ digits of π
- This is the $m = 163$ case of the following:

THM Chudnovskys (1993) For $\tau = (1 + \sqrt{-m})/2$,

$$\frac{1}{\pi} = \sqrt{\frac{m(J(\tau)-1)}{J(\tau)}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)! n!^3} \frac{(1 - s_2(\tau))/6 + n}{(1728 J(\tau))^n},$$

where

$$J(\tau) = \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}, \quad s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)} \left(E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau} \right).$$

Review: Singular moduli

FACT If f is a modular function and τ_0 a quadratic irrationality, then $f(\tau_0)$ is an algebraic number.

- $A \cdot \tau_0 = \frac{a\tau_0+b}{c\tau_0+d} = \tau_0$ for some $A \in \mathrm{GL}_2(\mathbb{Z})$
- Modular equation: $P(f(A \cdot \tau), f(\tau)) = 0$
- $Q(f(\tau_0)) = 0$ where $Q(x) = P(x, x)$

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Trouble: Complexity of modular equation increases extremely quickly.

EG Sometimes we can do much better.

For D odd let $\tau_D = \frac{1}{2}(1+\sqrt{D})$ [else $\tau_D = \frac{1}{2}\sqrt{D}$]. Let $\mathfrak{Q}_D$ be the primitive positive definite binary quadratic forms of discriminant D. For $Q \in \mathfrak{Q}_D$ let τ_Q be the root of $Q(\tau,1)=0$.

Then the conjugates of $j(\tau_D)$ are given by $j(\tau_Q)$, $Q \in \mathfrak{Q}_D$. In particular, these are algebraic numbers of degree $h(D)$.

The Chudnovsky series, again

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- $\mathbb{Q}(\sqrt{-163})$ has class number one.

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- $\mathbb{Q}(\sqrt{-163})$ has class number one.
- Current world record:
  5 trillion digits of π by Kondo and Yee

Notation

- Eisenstein series of weight 2:

$$E_2(\tau) = 1 - 24 \sum_{n \geqslant 1} \frac{n\, e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}}$$

- Standard Jacobi theta functions:

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 \tau}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 \tau}$$

- Elliptic modulus $k(\tau)$ and complementary modulus $k'(\tau)$:

$$k(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)} \right)^2, \qquad k'(\tau) = \left(\frac{\theta_4(\tau)}{\theta_3(\tau)} \right)^2$$

- Complete elliptic integral $K(k)$ of the first kind:

$$\frac{2}{\pi} K\left(k(\tau)\right) = {}_2F_1\left(\left. \begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \right| k^2(\tau) \right) = \theta_3(\tau)^2$$

General form of Ramanujan-type series for $1/\pi$

$$\frac{1}{\pi} = \alpha \sum_{n=0}^{\infty} a_n (A + Bn) \lambda^n$$

- α an algebraic number
- A, B, λ preferably rational numbers
- a_n a rational sequence

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  Typically, there is a modular function $x(\tau)$ and a modular form $f(\tau)$ such that

$$f(\tau) = \sum_{n=0}^{\infty} a_n x(\tau)^n.$$

In particular, the sequence a_n usually satisfies a linear recurrence.

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$$f(\tau) = \sum_{n=0}^{\infty} a_n x(\tau)^n.$$

EG If $a_n = \frac{(1/2)_n^3}{n!^3}$ then

$$\sum_{n=0}^{\infty} a_n x^n = {}_3F_2\left(\left.\begin{matrix}1/2, 1/2, 1/2\\ 1, 1\end{matrix}\right| x\right) = {}_2F_1\left(\left.\begin{matrix}1/2, 1/2\\ 1\end{matrix}\right| t\right)^2$$

with $x = 4t(1-t)$. Thus, here,

$$x(\tau) = 4k^2(\tau)(1-k^2(\tau)), \qquad f(\tau) = \theta_3(\tau)^4.$$

- For Sun's $520/\pi$ series, we will see a slight variation on this theme.

CONJ

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- Introduce:

$$A(x,y) = \sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k y^{2k-n}$$

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$$233A\left(\frac{1}{480}, 8\right) + 1054\,(\theta_x A)\left(\frac{1}{480}, 8\right) = \frac{520}{\pi}$$

- Here, $\theta_x = x\frac{\mathrm{d}}{\mathrm{d}x}$.

A crucial result of Wan and Zudilin

- After some manipulation and a hypergeometric transformation:

$$A(x,y) = \sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k y^{2k-n}$$
$$= \sum_{k=0}^{\infty} (-xy)^k \binom{2k}{k}^2 P_{2k}\left(\sqrt{1+\frac{4x}{y}}\right)$$

For $(x, y) = \left(\frac{1}{480}, 8\right)$ convergence is geometric with ratio $-\frac{64}{225}$.

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THM Wan Zudilin (2012)

When X and Y lie in a certain neighborhood of 1, then

$$\sum_{k=0}^{\infty} \left(\frac{X-Y}{4(1+XY)}\right)^{2k} \binom{2k}{k}^2 P_{2k}\left(\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right)$$
$$= \frac{1+XY}{2} {}_2F_1\left(\left.\begin{matrix}1/2, 1/2\\ 1\end{matrix}\right| 1-X^2\right) {}_2F_1\left(\left.\begin{matrix}1/2, 1/2\\ 1\end{matrix}\right| 1-Y^2\right).$$

Modular parametrization

- For **appropriate** x, y and X, Y,

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  provided that

$$-xy = \left(\frac{X-Y}{4(1+XY)}\right)^2, \qquad 1+\frac{4x}{y} = \left[\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right]^2. \qquad (*)$$

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LEM Let $(x,y) = (\frac{1}{480}, 8)$. If $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ and

$$X = k'(\tau_0), \qquad Y = k'(5\tau_0),$$

then (☺) holds in a neighborhood of the given values.

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then (☺) holds in a neighborhood of the given values.

- If $\tau_1 = -\frac{1}{10\tau_0}$ then $X = k'(\tau_1)$, $Y = k'(5\tau_1)$ satisfy (*) but not (☺).

Our singular moduli

FACT If f is a modular function and τ a quadratic irrationality, then $f(\tau)$ is an algebraic number.

- Here, $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ and

$$X = k'(\tau_0) \approx 0.57884718 - 0.81543604i,$$
$$Y = k'(5\tau_0) \approx 0.99999998 - 0.00021224i.$$

- X and Y both have minimal polynomial $z^8 p\left(z^2 + 1/z^2\right)$ where

$$p(z) = z^4 + 88796296z^3 + 237562136z^2 - 595063264z - 470492144.$$

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- In fact:

$$X = i\left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}}\right)^4$$

$$Y = i\left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} + \sqrt{\frac{1-\sqrt{5}}{2}}\right)^4$$

Final setup

CONJ

$$233A\left(\frac{1}{480},8\right)+1054\,(\theta_x A)\left(\frac{1}{480},8\right)=\frac{520}{\pi}$$

- We will now employ the notations:

$$F(\alpha)={}_2F_1\left(\begin{matrix}1/2,1/2\\1\end{matrix}\middle|\alpha\right) \qquad \alpha=1-X^2=k^2(\tau_0)$$

$$G(\alpha)=\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}F(\alpha) \qquad \beta=1-Y^2=k^2(5\tau_0)$$

- Then: $A(x,y)=\frac{1+XY}{2}F(1-X^2)F(1-Y^2)$

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CONJ

$$r_1F(\alpha)F(\beta)+r_2G(\alpha)\,F(\beta)+r_3F(\alpha)G(\beta)=\frac{520}{\pi}$$

- Here, and in the sequel, r_i (and later s_i,t_i) denote algebraic numbers.

  Whose value may change upon reuse.

Several reductions

CONJ

$$r_1 F(\alpha)F(\beta) + r_2 G(\alpha) F(\beta) + r_3 F(\alpha) G(\beta) = \frac{520}{\pi}$$

- $\beta = k^2(5\tau)$ has degree 5 over $\alpha = k^2(\tau)$.
  Using Ramanujan's expression for the multiplier $\frac{F(\alpha)}{F(\beta)}$:

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$$s_1 F(\alpha)^2 + s_2 F(\alpha) G(\alpha) = \frac{520}{\pi}$$

- Writing $G(\alpha)$ in terms of the weight 3 quasi-modular form $\frac{\mathrm{d}}{\mathrm{d}\tau} F(\alpha)$:

CONJ

$$t_1 F(\alpha)^2 + 52\sqrt{5} E_2(\tau_0) = \frac{520}{\pi}$$

Values of the Eisenstein series E_2

FACT Let τ_* be a quadratic irrationality and f a weight 2 modular form. Then $E_2(\tau_*) = \frac{r_1}{\pi} + r_2 f(\tau_*)$.

- Follows from:
  - $\dfrac{N E_2(N\tau) - E_2(\tau)}{f(\tau)}$ is a modular function.
  - $E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \dfrac{6\tau}{\pi i}$
  - If $\tau = i/\sqrt{N}$ then $-1/\tau = i/\sqrt{N} = N\tau$.

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  - If $\tau = i/\sqrt{N}$ then $-1/\tau = i/\sqrt{N} = N\tau$.
- Our interest is in $f(\tau) = \theta_3(\tau)^4 = F(\alpha)^2$.

  Unfortunately, rigorous computation of the algebraic numbers r_1, r_2 is, at best, tedious and relies heavily on modular equations tabulated by Ramanujan and proved by Andrews and Berndt.

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EG Ramanujan's multiplier of the second kind:

$$R_p\left(l, k\right) := \frac{pE_2(p\tau) - E_2(\tau)}{\theta_3^2(p\tau)\theta_3^2(\tau)}$$

is an algebraic function of $l := k(p\tau)$ and $k := k(\tau)$.

$$R_2\left(l, k\right) = l' + k$$

$$R_3\left(l, k\right) = 1 + kl + k'l'$$

$$R_5\left(l, k\right) = (3 + kl + k'l')\sqrt{\frac{1 + kl + k'l'}{2}}$$

Challenge answered

CONJ

$$t_1 F(\alpha)^2 + 52\sqrt{5} E_2(\tau_0) = \frac{520}{\pi}$$

- We just saw: $E_2(\tau_0) = \frac{2\sqrt{5}}{\pi} + s_2 F(\alpha)^2$

Challenge answered

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$$t_1 F(\alpha)^2 + 52\sqrt{5} E_2(\tau_0) = \frac{520}{\pi}$$

- We just saw: $E_2(\tau_0) = \frac{2\sqrt{5}}{\pi} + s_2 F(\alpha)^2$
- After verifying that $t_1 + 52\sqrt{5} s_2 = 0$:

THM S-Rogers (2012) Sun's conjecture is true:

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

Two open problems

- Guillera found (and in several cases proved) Ramanujan-type series for $1/\pi^2$ such as

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2}.$$

For the proven series only WZ style proofs exist.

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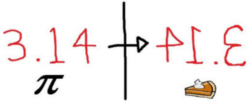
  For the proven series only WZ style proofs exist.
- As observed by L. Van Hamme, many series for $1/\pi$ have (mostly conjectural) p-analogues. In our case: (Sun, 2011)

$$\sum_{n=0}^{\infty} \frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k 8^{2k-n} = \frac{520}{\pi}$$

$$\sum_{n=0}^{p-1} \frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k 8^{2k-n} \overset{?}{\equiv} p\left(\frac{-1}{p}\right)\left(221+12\left(\frac{p}{15}\right)\right) \mod p^2$$

THANK YOU!

Happy Pi Day!



- Slides for this talk will be available from my website: `http://arminstraub.com/talks`

Mathew D. Rogers, Armin Straub
A solution of Sun's $\$520$ *challenge concerning* $\frac{520}{\pi}$
Int. Journal of Number Theory (to appear)

Illustration taken from: `http://www.geekologie.com/2010/04/omg-omg-omg-314-in-a-mirror-sp.php`