A solution of Sun's $\$520$ challenge concerning $\frac{520}{\pi}$

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Symbolic Computation and Special Functions

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Based on joint work with:

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CONJ

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- roughly, each two terms of the outer sum give one correct digit

> *I would like to offer \$520 (520 US dollars) for the person who could give the first correct proof of (*) in 2012 because May 20 is the day for Nanjing University.*
>
> **Zhi-Wei Sun** (2011)

The earliest series for $1/\pi$?

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1.3}{2.4}\right)^3 - 13\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(-1)^n(4n+1)$$

- Included in first letter of Ramanujan to Hardy

  but already given by Bauer in 1859 and further studied by Glaisher

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  but already given by Bauer in 1859 and further studied by Glaisher
- Limiting case of the terminating (Zeilberger, 1994)

$$\frac{\Gamma(3/2+m)}{\Gamma(3/2)\Gamma(m+1)} = \sum_{n=0}^{\infty} \frac{(1/2)_n^2(-m)_n}{n!^2(3/2+m)_n}(-1)^n(4n+1)$$

which has a WZ proof

Carlson's theorem justifies setting $m = -1/2$.

The first of Ramanujan's series

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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- Starred in High School Musical, a 2006 Disney production
- Both series also have WZ proof (Guillera, 2006)

  but no such proof known for more general series

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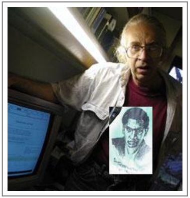
Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$



- Instead of proof, Ramanujan hints at "corresponding theories" which he unfortunately never developed
- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

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  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.
- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

The Chudnovsky series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(3n)! n!^3} \frac{13591409 + 545140134n}{640320^{3n+3/2}}$$

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- Used by David and Gregory Chudnovsky in 1988 to compute $2,260,331,336$ digits of π
- This is the $m = 163$ case of the following:

THM **Chud-novskys (1993)** For $\tau = (1 + \sqrt{-m})/2$,

$$\frac{1}{\pi} = \sqrt{\frac{m(J(\tau) - 1)}{J(\tau)}} \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)! n!^3} \frac{(1 - s_2(\tau))/6 + n}{(1728 J(\tau))^n},$$

where

$$J(\tau) = \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)}, \quad s_2(\tau) = \frac{E_4(\tau)}{E_6(\tau)} \left(E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau} \right).$$

Review: Singular moduli

FACT f a modular function, τ_0 a quadratic irrationality
$\Longrightarrow \quad f(\tau_0)$ is an algebraic number.

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- Such τ_0 is **fixed** by some $A \in \mathrm{GL}_2(\mathbb{Z})$:

$$A \cdot \tau_0 = \frac{a\tau_0 + b}{c\tau_0 + d} = \tau_0$$

- Two modular functions are related by a **modular equation**:

$$P(f(A \cdot \tau), f(\tau)) = 0$$

- Hence: $Q(f(\tau_0)) = 0$ where $Q(x) = P(x, x)$

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Trouble: Complexity of modular equation increases very quickly.

Review: Values of the j-function

- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$
- **Modular polynomial** $\Phi_N \in \mathbb{Z}[x, y]$ such that $\Phi_N(j(N\tau), j(\tau)) = 0$.

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RK Φ_N is $O(N^3 \log N)$ bits.

EG

$$\begin{aligned}\Phi_2(x,y) = {} & x^3 + y^3 - x^2y^2 + 2^4 \cdot 3 \cdot 31(x^2 + xy^2) \\ & - 2^4 \cdot 3^4 \cdot 5^3(x^2 + y^2) + 3^4 \cdot 5^3 \cdot 4027xy \\ & + 2^8 \cdot 3^7 \cdot 5^6(x + y) - 2^{12} \cdot 3^9 \cdot 5^9\end{aligned}$$

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$$\begin{aligned}\Phi_{11}(x, y) = {} & x^{12} + y^{12}x^{11}y^{11} + 8184x^{11}y^{10} - 28278756x^{11}y^9 \\ & + \ldots \text{several pages} \ldots + \\ & + 3924233450945276549086996 \ldots \text{100 digits} \ldots 000\end{aligned}$$

$\Phi_{11}(x, y)$ due to Kaltofen–Yui, 1984.

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$\Phi_{11}(x,y)$ due to Kaltofen–Yui, 1984.
$\Phi_{10007}(x,y)$ would require an estimated 4.8TB.

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Options for computation of singular moduli:

- via modular equations
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EG Let us evaluate $j(\frac{1+\sqrt{-23}}{2})$.

CFT: The Galois conjugates are $j(\frac{1+\sqrt{-23}}{4})$, $j(\frac{-1+\sqrt{-23}}{4})$.

$$\left(x - j(\tfrac{1+\sqrt{-23}}{2})\right)\left(x - j(\tfrac{1+\sqrt{-23}}{4})\right)\left(x - j(\tfrac{-1+\sqrt{-23}}{4})\right)$$
$$= x^3 + 3491750x^2 - 5151296875x + 12771880859375$$

Degree is $h(-23) = 3$.

THM Chudnovskys (1993)

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- $\mathbb{Q}(\sqrt{-163})$ has class number one.

The Chudnovsky series, revisited

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- $\mathbb{Q}(\sqrt{-163})$ has class number one.
- Current world record:
  10 trillion digits of π
  by Shigeru Kondo and Alexander Yee
  on a self-built desktop pc in 191 days

Notation

- Eisenstein series of weight 2:

$$E_2(\tau) = 1 - 24 \sum_{n \geqslant 1} \frac{n\, e^{2\pi i n \tau}}{1 - e^{2\pi i n \tau}}$$

- Standard Jacobi theta functions:

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+1/2)^2 \tau}, \quad \theta_3(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \theta_4(\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i n^2 \tau}$$

- Elliptic modulus $k(\tau)$ and complementary modulus $k'(\tau)$:

$$k(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)} \right)^2, \qquad k'(\tau) = \left(\frac{\theta_4(\tau)}{\theta_3(\tau)} \right)^2$$

- Complete elliptic integral $K(k)$ of the first kind:

$$\frac{2}{\pi} K\left(k(\tau)\right) = {}_2F_1 \left(\left. \begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \right| k^2(\tau) \right) = \theta_3(\tau)^2$$

General form of Ramanujan-type series for $1/\pi$

$$\frac{1}{\pi} = \alpha \sum_{n=0}^{\infty} a_n (A + Bn) \lambda^n$$

- α an algebraic number
- A, B, λ preferably rational numbers
- a_n a rational sequence

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  Typically, there is a modular function $x(\tau)$ and a modular form $f(\tau)$ such that

$$f(\tau) = \sum_{n=0}^{\infty} a_n x(\tau)^n.$$

In particular, the sequence a_n usually satisfies a linear recurrence.

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EG If $a_n = \frac{(1/2)_n^3}{n!^3}$ then

$$\sum_{n=0}^{\infty} a_n x^n = {}_3F_2\left(\left. \begin{matrix} 1/2, 1/2, 1/2 \\ 1, 1 \end{matrix} \right| x \right) = {}_2F_1\left(\left. \begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \right| t \right)^2$$

with $x = 4t(1-t)$. Thus, here,

$$x(\tau) = 4k^2(\tau)(1 - k^2(\tau)), \qquad f(\tau) = \theta_3(\tau)^4.$$

- For Sun's $520/\pi$ series, we have a slight variation on this theme.

The modular explanation of series for $1/\pi$

$$\frac{1}{\pi} = \alpha \sum_{n=0}^{\infty} a_n (A + Bn) \lambda^n \qquad (1/\pi)$$

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- Modular function $x(\tau)$ and modular form $f(\tau)$ such that

$$f(\tau) = \sum_{n=0}^{\infty} a_n x(\tau)^n.$$

- If $x(\tau_0) = \lambda$, then

$$\sum_{n=0}^{\infty} a_n (A + Bn) \lambda^n = A f(\tau_0) + \lambda B \frac{f'(\tau_0)}{x'(\tau_0)}.$$

- $f'(\tau)$ is a **quasimodular** form.

Review: Quasimodular forms

- The ring $\widetilde{M}_*(\Gamma)$ of quasimodular forms is the differential closure of the ring of modular forms $M_*(\Gamma)$. $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ of finite index

THM Kaneko–Zagier, 1995

Let E_2 be the weight 2 Eisenstein series. Then:

$$\widetilde{M}_*(\Gamma) = M_*(\Gamma) \otimes \mathbb{C}[E_2]$$

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$$\Longrightarrow \quad E_2(\tau_0) = \frac{r_1}{\pi} + r_2 f(\tau_0)$$

r_1, r_2 algebraic numbers

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proof

- $\dfrac{N E_2(N\tau) - E_2(\tau)}{f(\tau)}$ is a modular function.
- $E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \dfrac{6\tau}{\pi i}$
- If $\tau = i/\sqrt{N}$ then $-1/\tau = i/\sqrt{N} = N\tau$.

□

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- Our interest is in $f(\tau) = \theta_3(\tau)^4$.

  Unfortunately, rigorous computation of the algebraic numbers r_1, r_2 is, at best, tedious and relies heavily on modular equations tabulated by Ramanujan and proved by Andrews and Berndt.

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EG Ramanujan's multiplier of the second kind:

$$R_N(l,k) := \frac{N E_2(N\tau) - E_2(\tau)}{\theta_3^2(N\tau)\theta_3^2(\tau)}$$

is an algebraic function of $l := k(N\tau)$ and $k := k(\tau)$.

$$R_2(l,k) = l' + k$$

$$R_3(l,k) = 1 + kl + k'l'$$

$$R_5(l,k) = (3 + kl + k'l')\sqrt{\frac{1 + kl + k'l'}{2}}$$

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- If τ is a quadratic irrationality, then algebraic A, B exist such that:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B x(\tau) \frac{f'(\tau)}{x'(\tau)} = \frac{1}{\pi}$$

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Main practical issues:

- identifying involved modular parametrization
- rigorous computation of values of modular functions and combinations of quasimodular forms

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Next: modular parametrization for Sun's series

Sun's series for $520/\pi$

CONJ

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- Introduce:

$$A(x,y) = \sum_{n=0}^{\infty} x^n \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k y^{2k-n}$$

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CONJ

$$233 A\left(\frac{1}{480}, 8\right) + 1054 \left(\theta_x A\right)\left(\frac{1}{480}, 8\right) = \frac{520}{\pi}$$

- Here, $\theta_x = x \frac{\mathrm{d}}{\mathrm{d}x}$.

A crucial result of Wan and Zudilin

- After some manipulation and a hypergeometric transformation:

$$A(x,y)=\sum_{n=0}^{\infty}x^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k y^{2k-n}$$
$$=\sum_{k=0}^{\infty}(-xy)^k\binom{2k}{k}^2 P_{2k}\left(\sqrt{1+\frac{4x}{y}}\right)$$

For $(x,y)=\left(\frac{1}{480},8\right)$ convergence is geometric with ratio $-\frac{64}{225}$.

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THM Wan Zudilin (2012) When X and Y lie in a certain neighborhood of 1, then

$$\sum_{k=0}^{\infty}\left(\frac{X-Y}{4(1+XY)}\right)^{2k}\binom{2k}{k}^2 P_{2k}\left(\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right)$$
$$=\frac{1+XY}{2}\,{}_2F_1\left(\left.\begin{matrix}1/2,1/2\\1\end{matrix}\right|1-X^2\right){}_2F_1\left(\left.\begin{matrix}1/2,1/2\\1\end{matrix}\right|1-Y^2\right).$$

Modular parametrization

- For **appropriate** x, y and X, Y,

$$A(x,y) = \frac{1+XY}{2} {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| 1-X^2\right) {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| 1-Y^2\right) \quad (☺)$$

provided that

$$-xy = \left(\frac{X-Y}{4(1+XY)}\right)^2, \qquad 1+\frac{4x}{y} = \left[\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right]^2. \qquad (*)$$

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LEM Let $(x,y) = (\frac{1}{480}, 8)$. If $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ and

$$X = k'(\tau_0), \qquad Y = k'(5\tau_0),$$

then (☺) holds in a neighborhood of the given values.

Modular parametrization

- For **appropriate** x, y and X, Y,

$$A(x,y) = \frac{1+XY}{2} {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| 1-X^2\right) {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| 1-Y^2\right) \qquad (☺)$$

provided that

$$-xy = \left(\frac{X-Y}{4(1+XY)}\right)^2, \qquad 1 + \frac{4x}{y} = \left[\frac{(X+Y)(1-XY)}{(X-Y)(1+XY)}\right]^2. \qquad (*)$$

LEM Let $(x,y) = (\frac{1}{480}, 8)$. If $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ and

$$X = k'(\tau_0), \qquad Y = k'(5\tau_0),$$

then (☺) holds in a neighborhood of the given values.

- If $\tau_1 = -\frac{1}{10\tau_0}$ then $X = k'(\tau_1)$, $Y = k'(5\tau_1)$ satisfy (*) but not (☺).

Our singular moduli

FACT f a modular function, τ_0 a quadratic irrationality
$\implies$ $f(\tau_0)$ is an algebraic number.

- Here, $\tau_0 = \frac{1}{2} + \frac{3}{10}\sqrt{-5}$ and

$$X = k'(\tau_0) \approx 0.57884718 - 0.81543604i,$$
$$Y = k'(5\tau_0) \approx 0.99999998 - 0.00021224i.$$

- X and Y both have minimal polynomial $z^8 p\left(z^2 + 1/z^2\right)$ where

$$p(z) = z^4 + 88796296z^3 + 237562136z^2 - 595063264z - 470492144.$$

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- In fact:

$$X = i\left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}}\right)^4$$

$$Y = i\left(\sqrt{\frac{7-3\sqrt{5}}{4}} - \sqrt{\frac{3-3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3-\sqrt{5}}{2}} + \sqrt{\frac{1-\sqrt{5}}{2}}\right)^4$$

Sun's challenge answered

Modulo plenty of computation, we are now in a position to prove:

THM S-Rogers (2012) Sun's conjecture is true.

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

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Sun conjectured a total of 17 series of the above shape, such as

$$\frac{35\sqrt{6}}{4\pi} = \sum_{n=0}^{\infty} \frac{19n + 3}{240^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} 6^{2k-n}.$$

They follow in the same way.

Computer algebra challenges

- Devise fast and rigorous methods to compute singular moduli
  - for instance, for modular functions built from eta quotients
- Automatize computations with quasimodular forms such as
  - representing (certain classes of) quasimodular forms as polynomials in E_2 with modular coefficients
  - relating values of quasimodular forms at CM points to values of modular forms at CM points

Two open problems

- Guillera found (and in several cases proved) Ramanujan-type series for $1/\pi^2$ such as

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2}.$$

For the proven series only WZ style proofs exist.

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- As observed by van Hamme, many series for $1/\pi$ have (mostly conjectural) p-analogues. In our case: (Sun, 2011)

$$\sum_{n=0}^{\infty} \frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k 8^{2k-n} = \frac{520}{\pi}$$

$$\sum_{n=0}^{p-1} \frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k 8^{2k-n} \stackrel{?}{=} p\left(\frac{-1}{p}\right)\left(221+12\left(\frac{p}{15}\right)\right) \mod p^2$$

THANK YOU!

- Slides for this talk will be available from my website:
  `http://arminstraub.com/talks`

Mathew D. Rogers, Armin Straub
A solution of Sun's \$520 *challenge concerning* $\frac{520}{\pi}$
Int. Journal of Number Theory (Vol. 9, Nr. 5, 2013, p. 1273-1288)