Apéry numbers and their experimental siblings

Challenges in 21st Century Experimental Mathematical Computation
ICEM, Brown University

Armin Straub

July 22, 2014

University of Illinois at Urbana–Champaign

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Jon Borwein   Dirk Nuyens   James Wan   Wadim Zudilin   Robert Osburn   Brundaban Sahu   Mathew Rogers

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

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THM
Apéry '78 $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. $\square$

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q
Beukers,
Zagier
Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
    - 4 hypergeometric and 4 Legendrian solutions
    - 6 sporadic solutions

- Similar (and intertwined) story for:
    - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
    - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

Apéry-like numbers

- Hypergeometric and Legendrian solutions have generating functions

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}, \alpha, 1-\alpha \\ 1, 1 \end{matrix} \middle| 4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix} \alpha, 1-\alpha \\ 1 \end{matrix} \middle| \frac{-C_\alpha z}{1-C_\alpha z}\right)^2,
$$

with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^{12n}}_{\text{modular function}}.$$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:
  $f(\tau)$ modular form of weight k
  $x(\tau)$ modular function
  $y(x)$ such that $y(x(\tau)) = f(\tau)$

  Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,

$$A(p) \equiv 5 \mod p^3.$$

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THM
Beukers,
Coster
'85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

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EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$A(5 \cdot 11^3) = 12488301\ldots$about 2000 digits$\ldots$about 8000 digits$\ldots 795652125$

Weirdly, with this wrong value, one still has

$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \mod 11^6.$$

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THM
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The Apéry numbers satisfy the **supercongruence** $\qquad (p \geqslant 5)$
$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}.$$

EG Simple combinatorics proves the congruence
$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \mod p^2.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,
$$\binom{2p}{p} \equiv 2 \mod p^3.$$

Supercongruences for Apéry-like numbers

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}$$

hold for all Apéry-like numbers.

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a, b, c)	$A(n)$	
$(7, 3, 81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11, 5, 125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(10, 4, 64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
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$(9, 3, -27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open
$(17, 5, 1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88

A generalization: multivariate supercongruences

THM
S 2013

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the **diagonal coefficients**.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

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- The Apéry numbers are the **diagonal coefficients**.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

- Both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

Short random walks
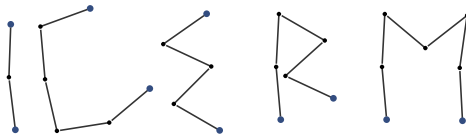
joint work with:

Jon Borwein
U. Newcastle, AU

Dirk Nuyens
K.U.Leuven, BE

James Wan
SUTD, SG

Wadim Zudilin
U. Newcastle, AU
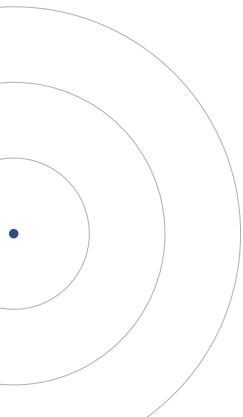
n steps in the plane
(length 1, random direction)

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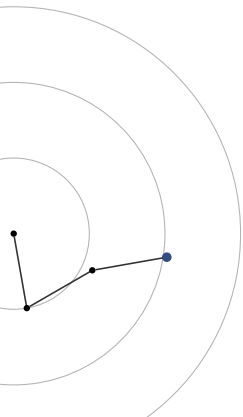
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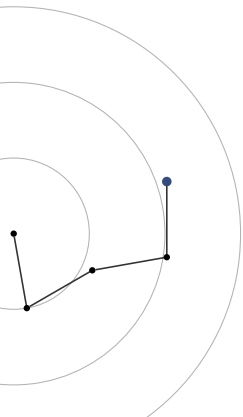
Random walks in the plane

n steps in the plane
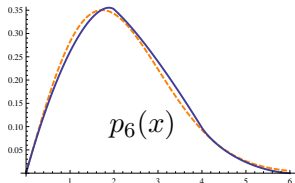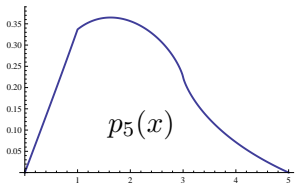(length 1, random direction)

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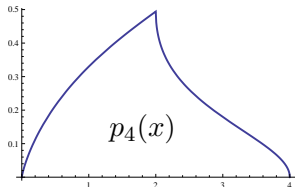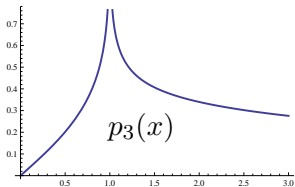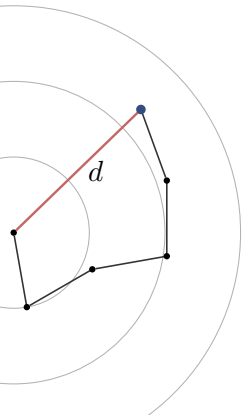
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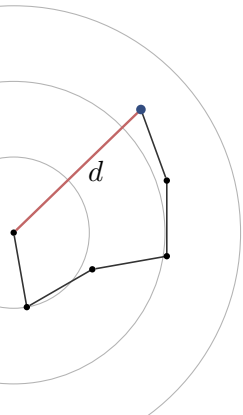(length 1, random direction)

Random walks in the plane

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- $p_n(x)$ — probability density of distance traveled

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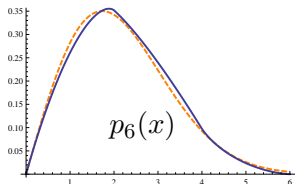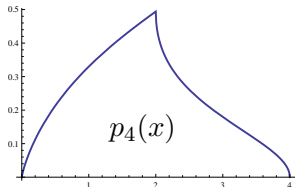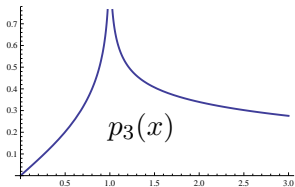
- $W_n(s) = \int_0^\infty x^s p_n(x)\, \mathrm{d}x$ — probability moments
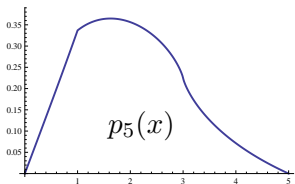
$$W_2(1) = \frac{4}{\pi}, \qquad W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

classical   Borwein–Nuyens–S–Wan, 2010

Moments of random walks

- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x) \, \mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j},$$

$$W_4(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

Moments of random walks

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$$W_3(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j},$$

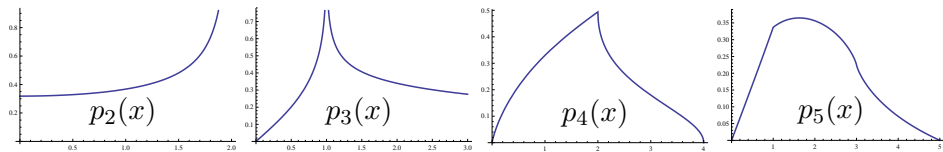$$W_4(2k) = \sum_{j=0}^k \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

> **THM**
> Borwein-
> Nuyens-
> S-Wan
> 2010
>
> $$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

Densities of random walks

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}} \qquad\qquad \text{easy}$$

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} \, _2F_1\left(\begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ 1 \end{array}\middle|\, \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right) \qquad \begin{array}{c} \text{classical} \\ \text{\small with a spin} \end{array}$$

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \, \text{Re} \, _3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{array}\middle|\, \frac{\left(16-x^2\right)^3}{108x^4}\right) \qquad \begin{array}{c} \text{new} \\ \text{\small BSWZ 2011} \end{array}$$

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4} \Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

Based on joint work with:

Mathew Rogers
(University of Montreal)

Ramanujan's series for $1/\pi$

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$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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$$\mathbf{\frac{16}{\pi}} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

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Srinivasa Ramanujan
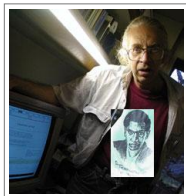*Modular equations and approximations to π*
Quart. J. Math., Vol. 45, p. 350–372, 1914

Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$

- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.

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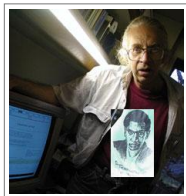
- First proof of all of Ramanujan's 17 series for $1/\pi$
  by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

Apéry-like numbers and series for $1/\pi$

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG
Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \displaystyle\sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

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- Sun offered a \$520 bounty for a proof the following series:

THM
Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n + 233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

Positivity of rational functions

$$\frac{1}{1 - (x + y + z) + 4xyz}$$

Based on joint work with:

Wadim Zudilin
(University of Newcastle)

Positivity of rational functions

CONJ Kauers– Zeilberger	$$\frac{1}{1 - (x + y + z + w) + 2(yzw + xzw + xyw + xyz) + 4xyzw}$$ has positive Taylor coefficients.

Positivity of rational functions

CONJ
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$$\frac{1}{1 - (x + y + z + w) + 2(yzw + xzw + xyw + xyz) + 4xyzw}$$

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PROP
S-Zudilin
2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

Positivity of rational functions

CONJ
Kauers–
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has positive Taylor coefficients.

PROP
S-Zudilin
2013

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$$d_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}^2.$$

- For such rational functions, should positivity be (essentially) implied by positivity of diagonal?   assuming positivity after setting one variable to zero

Summary and some open problems

- Apéry-like numbers are integer solutions to certain three-term recurrences
  - is the experimental list complete?
  - higher-order analogs, Calabi–Yau DEs
- Apéry-like numbers have interesting properties
  - modular parametrization; uniform explanation?
  - supercongruences; still open in several cases
- Apéry-like numbers occur in interesting places
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties
  - . . .

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

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