Properties and applications of Apéry-like numbers

International Conference on Algebra and Number Theory
Samsun, Turkey

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August 8, 2014

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Jon Borwein

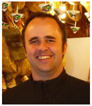
Dirk Nuyens

James Wan

Wadim Zudilin

Robert Osburn

Brundaban Sahu



Mathew Rogers

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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Q (Beukers, Zagier): Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.
- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\substack{\text{modular form} \\ 1+5q+13q^2+23q^3+O(q^4)}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\substack{\text{modular function} \\ q-12q^2+66q^3+O(q^4)}} .$$

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88: The Apéry numbers satisfy the **supercongruence** ($p \geqslant 5$)
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EG Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
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$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\}$$ "points of period n"

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

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CONJ
Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

THM
S 2013
The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

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THM **S 2013** The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

- Previously known: they are also the diagonal of Christol, '84

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to prove, but much harder to discover.

Apéry numbers as diagonals

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THM
S 2013

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- Univariate generating function:

$$\sum_{n \geq 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2 \left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right| - \frac{1024x}{(1 - x + z)^4} \right),$$

where $z = \sqrt{1 - 34x + x^2}$.

Apéry numbers as diagonals

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THM
S 2013

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically (pun intended) leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even}, \\ 5 \pmod 8, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

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Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the **diagonal coefficients**.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

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S 2013

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$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

- Both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

CONJ
S 2013

The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^3} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Franel numbers**

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

Short random walks

joint work with:

Jon Borwein
U. Newcastle, AU

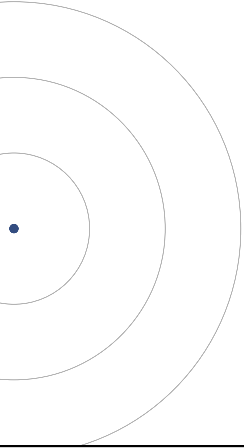
Dirk Nuyens
K.U.Leuven, BE

James Wan
SUTD, SG

Wadim Zudilin
U. Newcastle, AU

Random walks in the plane

n steps in the plane
(length 1, random direction)



What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

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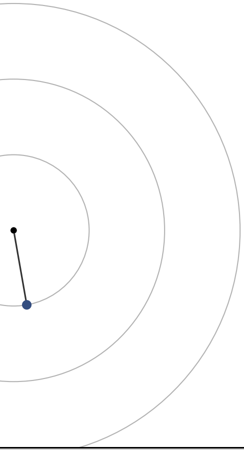

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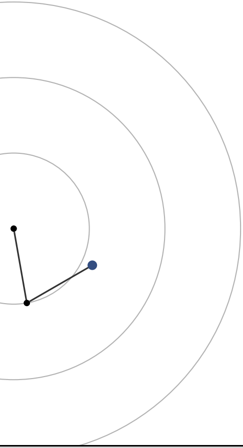

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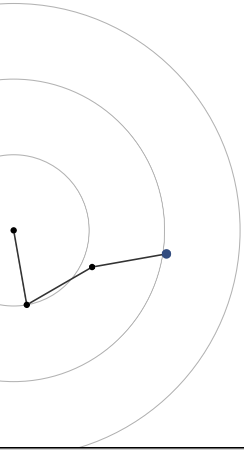

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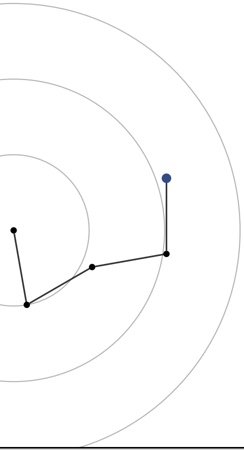

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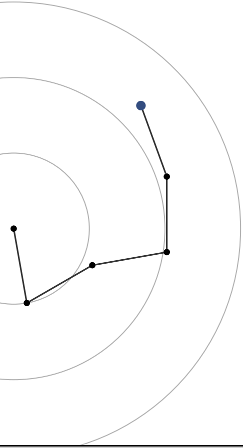

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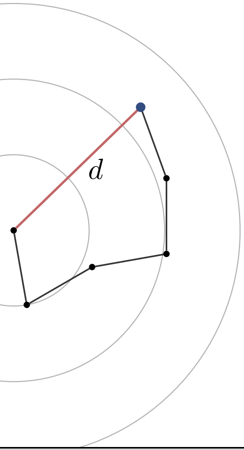

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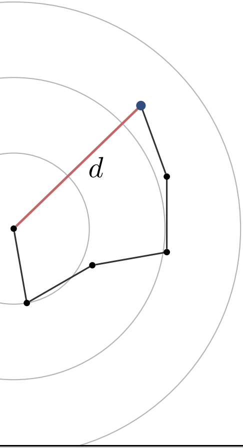

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$$\begin{aligned} p_n(x) &\quad \text{probability density} \\ W_n(s) &\quad s\text{th moment} \end{aligned}$$

EG

$$W_2(1) = \frac{4}{\pi}$$

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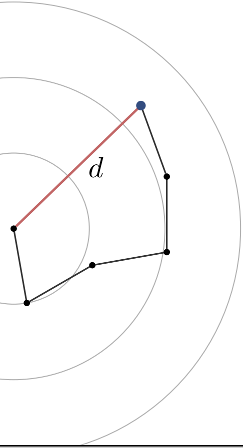

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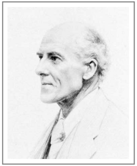

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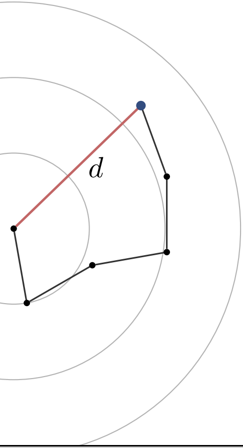

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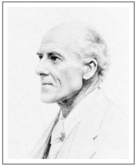

THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

Long random walks

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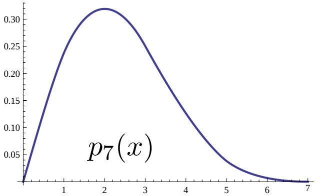

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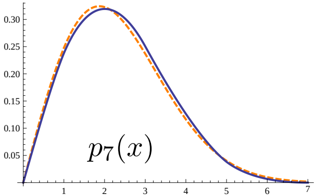

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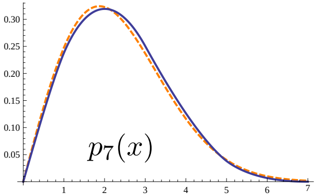
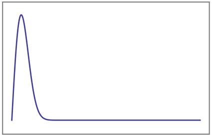


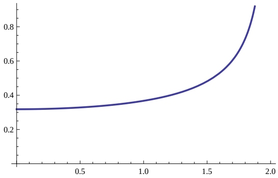
$$W_n(1) \approx \sqrt{n\pi}/2$$

> *The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*
>
> **Karl Pearson**, *1905*

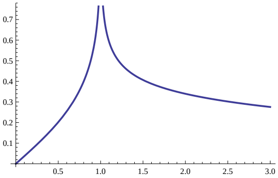


Densities of short walks

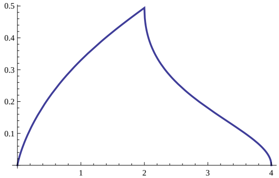
p_2



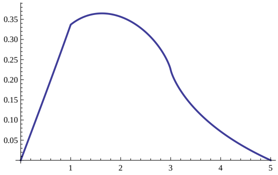
p_3



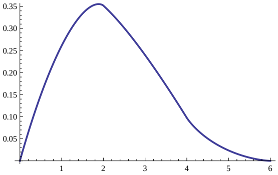
p_4



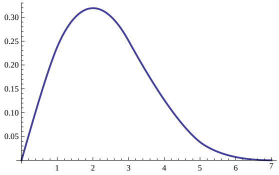
p_5



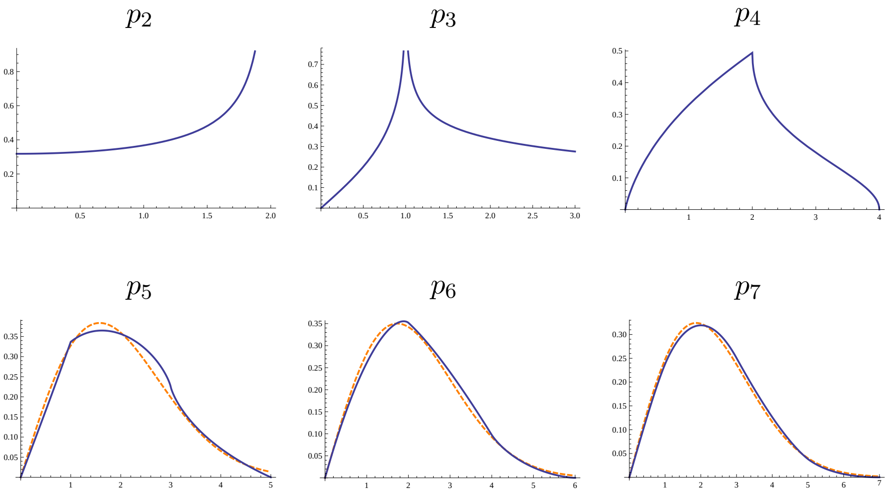
p_6



p_7



Densities of short walks



Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \ ??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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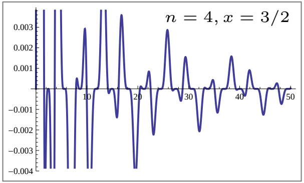

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \;?$$

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$$\left|1+e^{2\pi i y}\right| = \left|1+(\cos\pi y + i\sin\pi y)^2\right| = 2\cos(\pi y)$$

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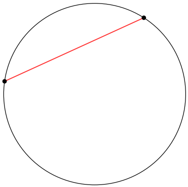
$$\begin{aligned} W_2(1) &= \int_0^1\int_0^1 \left|e^{2\pi ix}+e^{2\pi iy}\right| \mathrm{d}x\mathrm{d}y = \ ? \\ &= \int_0^1 \left|1+e^{2\pi iy}\right| \mathrm{d}y \\ &= \int_0^1 2\cos(\pi y)\mathrm{d}y \\ &= \frac{4}{\pi} \approx 1.27324 \end{aligned}$$

- This is the average length of a random arc on a unit circle.



The moments of random walks

DEF The sth moment $W_n(s)$ of the density p_n:

$$W_n(s) := \int_0^\infty x^s p_n(x)\,\mathrm{d}x = \int_{[0,1]^n} \left|e^{2\pi i x_1} + \ldots + e^{2\pi i x_n}\right|^s \mathrm{d}\boldsymbol{x}$$

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- On a desktop:

$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

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- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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n	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
5	2.008	5.000	14.29	45.00	152.3	545.0	2037.
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The even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

Apéry-like

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$

Domb numbers

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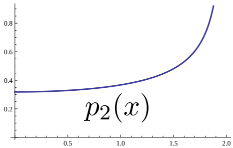
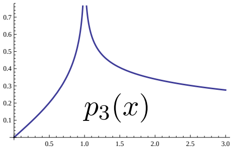
Domb numbers

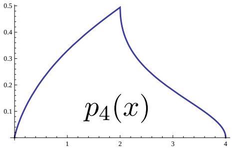
THM Borwein-Nuyens-S-Wan, 2010

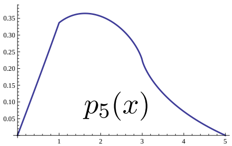
$$W_3(1) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \\ 1, 1 \end{matrix}\right| 4\right) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

Densities of random walks









$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi}\,\frac{x}{(3+x^2)}\,{}_2F_1\left(\begin{matrix}\frac13,\frac23\\ 1\end{matrix}\,\middle|\,\frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$ classical with a spin

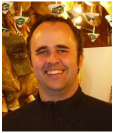
$$p_4(x) = \frac{2}{\pi^2}\,\frac{\sqrt{16-x^2}}{x}\,\mathrm{Re}\;{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac56,\frac76\end{matrix}\,\middle|\,\frac{\left(16-x^2\right)^3}{108x^4}\right)$$ new BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\,\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$

Based on joint work with:



Mathew Rogers
(University of Montreal)

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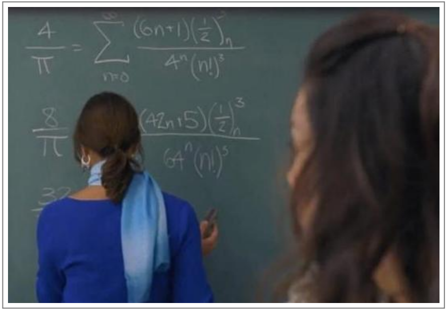
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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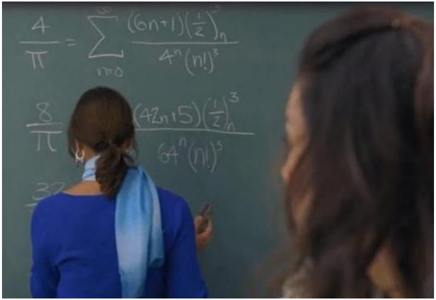
$$\frac{\mathbf{16}}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



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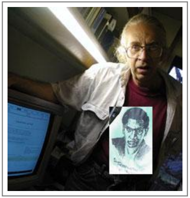
Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$



- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.

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- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG
Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG Chan-Chan-Liu 2003

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$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

- Sun offered a \$520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \dfrac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

Based on joint work with:

Wadim Zudilin
(University of Newcastle)

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

  is **positive** if $a_{n_1,\ldots,n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x,y,z) = \frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x,y,z) = \frac{1}{1-(x+y+z)+4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

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$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$.

CONJ Kauers-Zeilberger 2008 The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

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Recent proof of non-negativity by Scott and Sokal, 2013

PROP S-Zudilin 2013 The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

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THM S-Zudilin 2013

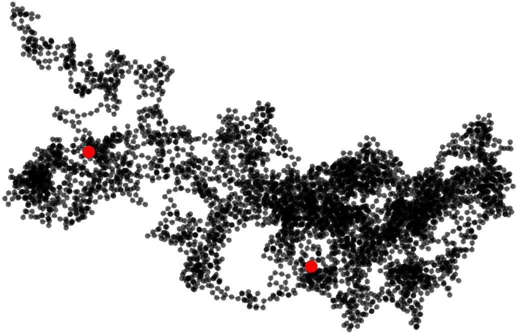
$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2 xy} \quad \text{is positive}$$

$\iff$ diagonal of F and $F|_{x_d=0}$ are positive

Summary and some open problems

- Apéry-like numbers are integer solutions to certain three-term recurrences
  - is the experimental list complete?
  - higher-order analogs, Calabi–Yau DEs
- Apéry-like numbers have interesting properties
  - modular parametrization; uniform explanation?
  - supercongruences; still open in several cases
- Apéry-like numbers occur in interesting places
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties
  - ...

Drunken birds



> *A drunk man will find his way home,*
> *but a drunk bird may get lost forever.*
> **Shizuo Kakutani**, *1911–2004*

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990