Properties and applications of Apéry-like numbers

Mathematics Colloquium
Tulane University

Armin Straub

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Jon Borwein

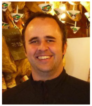
Dirk Nuyens

James Wan

Wadim Zudilin

Robert Osburn

Brundaban Sahu



Mathew Rogers

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

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THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Zagier's search and Apéry-like numbers

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Q Beukers, Zagier	Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

Apéry-like numbers

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle| 4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle| \frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha=2^4,3^3,2^6,2^4\cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k(-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k(-1)^k\binom{n}{k}^3\left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k\binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l}\binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k\binom{n}{k}^2\binom{n+k}{n}^2$

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i \tau}$

Modularity of Apéry-like numbers

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i \tau}$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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EG Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry-like numbers

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$	open
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$	Beukers, Coster '87-'88

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \operatorname{ct} \Lambda(x)^n$ — van Straten–Samol '09

  if origin is only interior pt of the Newton polyhedron of $\Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

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CONJ Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

Apéry numbers as diagonals

- Given a series

$$F(x_1,\ldots,x_d)=\sum_{n_1,\ldots,n_d\geqslant 0} a(n_1,\ldots,n_d)x_1^{n_1}\cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n,\ldots,n)$.

THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F\in K[[x_1,\ldots,x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1,\ldots,x_d)$.

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EG Christol 1984

The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to prove, but much harder to discover.

Apéry numbers as diagonals

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THM S 2013 The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Univariate generating function:

$$\sum_{n\geq 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

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  where $z=\sqrt{1-34x+x^2}$.
- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 & \pmod 8, \quad \text{if } n \text{ even},\\ 5 & \pmod 8, \quad \text{if } n \text{ odd}.\end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

Multivariate supercongruences

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Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

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- $\sum_{n \geqslant 0} a(n) x^n = F(x) \quad \Longrightarrow \quad \sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x)$ $\qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

Extended version

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Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^{\ell}_{>0}$ with $d = \lambda_1 + \ldots + \lambda_\ell$, and set $s(j) = \lambda_1 + \ldots + \lambda_{j-1}$. Define $A_\lambda(\boldsymbol{n})$ by

$$\left(\prod_{j=1}^{\ell} \left[1 - \sum_{r=1}^{\lambda_j} x_{s(j)+r} \right] - x_1 x_2 \cdots x_d \right)^{-1} = \sum_{\boldsymbol{n} \in \mathbb{Z}^d_{\geqslant 0}} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geq 2$, then, for all primes p and integers $r \geq 1$,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geq 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leq 2$, then, for primes $p \geq 5$ and integers $r \geq 1$,

$$A_\lambda(\boldsymbol{n}p^r) \equiv A_\lambda(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

Further examples

EG The Apéry-like numbers, associated with $\zeta(2)$,

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

are the diagonal coefficients of the rational function

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^3} B(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

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COR S 2014 We find

$$B(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3}{k},$$

and, for primes $p \geqslant 5$,

$$B(p^r\boldsymbol{n}) \equiv B(p^{r-1}\boldsymbol{n}) \pmod{p^{3r}}.$$

- The diagonal case recovers supercongruences of Coster, 1988.

Further examples

EG The numbers

$d = 3$: Franel, $d = 4$: Yang–Zudilin

$$Y_d(n) = \sum_{k=0}^{n} \binom{n}{k}^d,$$

are the diagonal coefficients of the rational function

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d) - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} Y_d(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

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COR S 2014 We find

$$Y_d(\boldsymbol{n}) = \sum_{k \geq 0} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_d}{k},$$

and, for $d \geqslant 2$ and primes $p \geqslant 5$,

$$Y_d(p^r \boldsymbol{n}) \equiv Y_d(p^{r-1}\boldsymbol{n}) \pmod{p^{3r}}.$$

- This generalizes a result of Chan–Cooper–Sica, 2010.

A conjectural multivariate supercongruence

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The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.

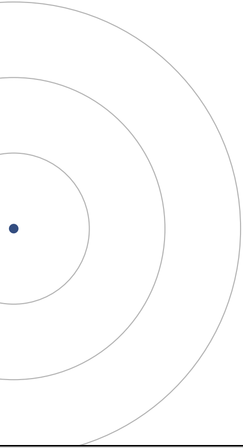
Short random walks

joint work with:

Jon Borwein	Dirk Nuyens	James Wan	Wadim Zudilin
U. Newcastle, AU	K.U.Leuven, BE	SUTD, SG	U. Newcastle, AU

Random walks in the plane

n steps in the plane
(length 1, random direction)



What is the distance traveled in n steps?

$p_n(x)$	probability density
$W_n(s)$	sth moment

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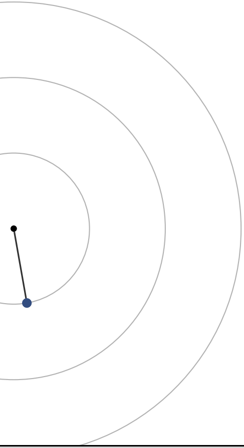

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$$p_n(x) \quad \text{probability density}$$
$$W_n(s) \quad s\text{th moment}$$

Random walks in the plane

n steps in the plane
(length 1, random direction)

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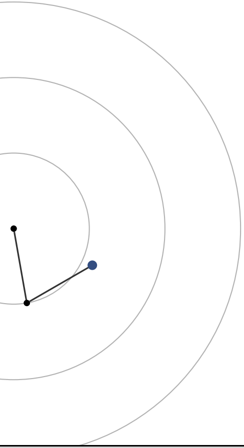

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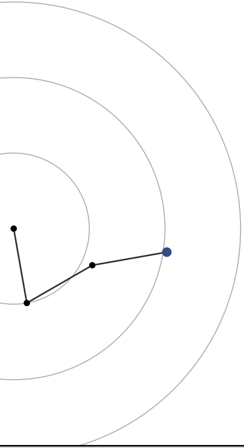

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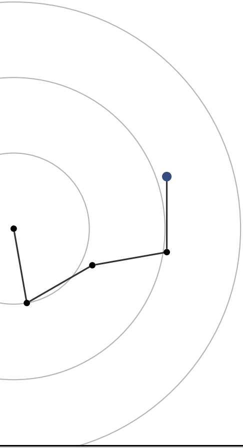

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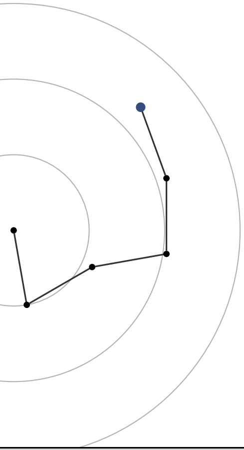

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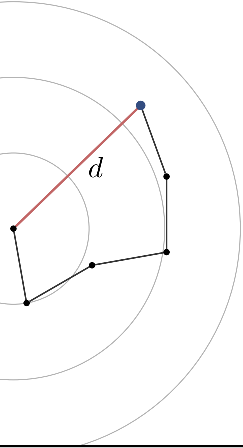

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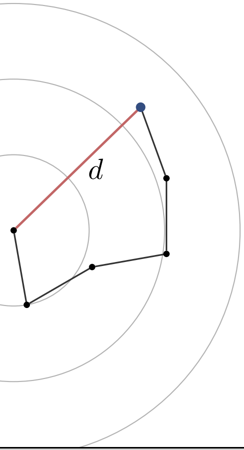

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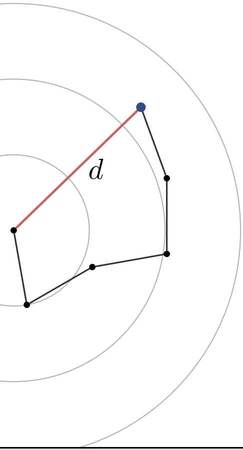

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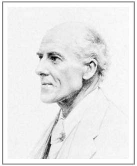

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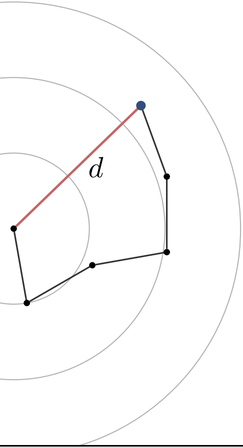

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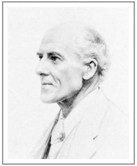

THM Rayleigh, 1905

$$p_n(x) \approx \frac{2x}{n} e^{-x^2/n} \qquad \text{for large } n$$

Long random walks

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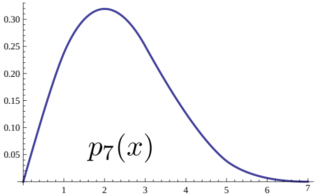

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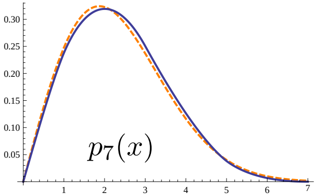

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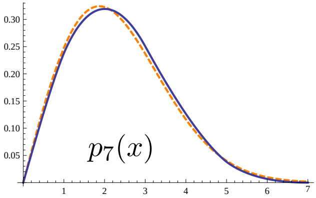
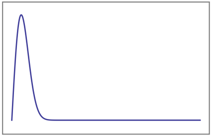
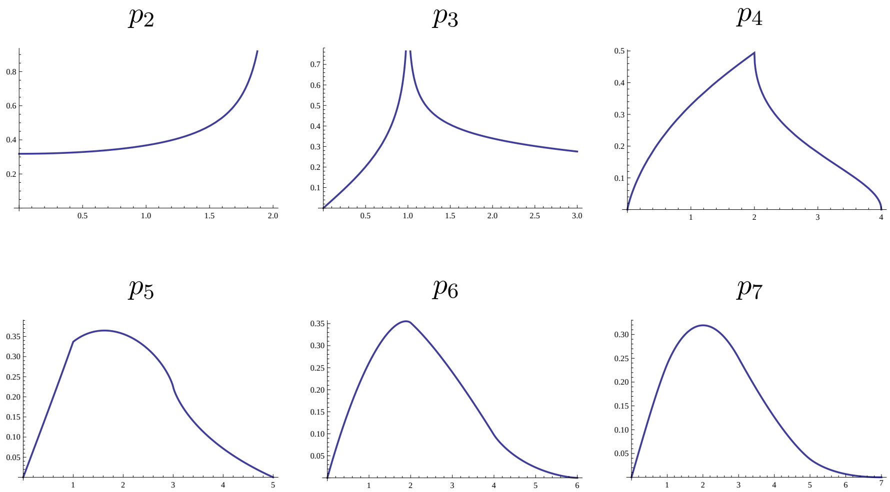


$$W_n(1) \approx \sqrt{n\pi}/2$$

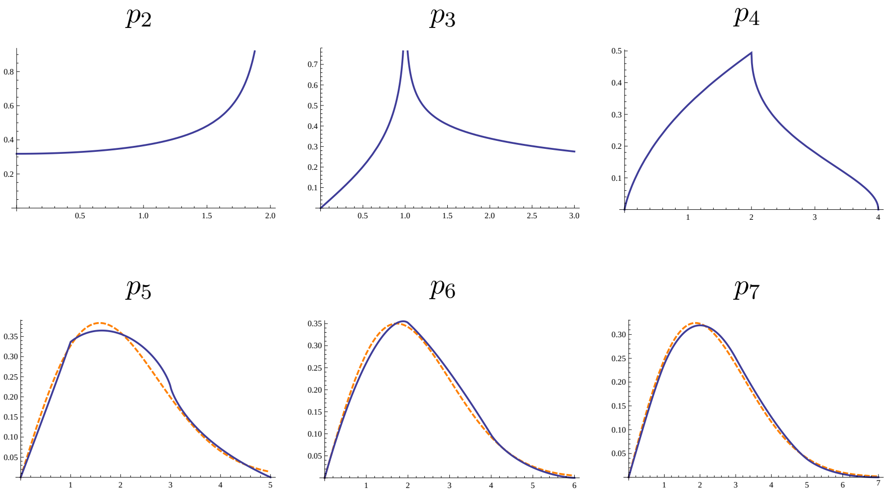
“*The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!*”

Karl Pearson, *1905*




p2
0.8
0.6
0.4
0.2
0.5 1.0 1.5 2.0
p3
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5 1.0 1.5 2.0 2.5 3.0
p4
0.5
0.4
0.3
0.2
0.1
1 2 3 4
p5
0.35
0.30
0.25
0.20
0.15
0.10
0.05
1 2 3 4 5
p6
0.35
0.30
0.25
0.20
0.15
0.10
0.05
1 2 3 4 5 6
p7
0.30
0.25
0.20
0.15
0.10
0.05
1 2 3 4 5 6 7

Densities of short walks



Classical results on the densities

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\,K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right)$$ G. J. Bennett 1905

$$p_4(x) = \,??$$

$$\vdots$$

$$p_n(x) = \int_0^\infty xtJ_0(xt)J_0^n(t)\,\mathrm{d}t$$ J. C. Kluyver 1906

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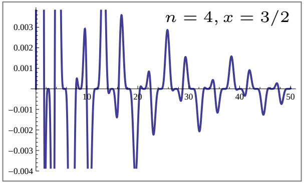

The average distance traveled in two steps

- The average distance in two steps:

$$W_2(1) = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x \mathrm{d}y = \ ?$$

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$$= \int_0^1 2\cos(\pi y)\mathrm{d}y$$

$$\left| 1 + e^{2\pi iy} \right|$$
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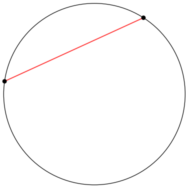
- This is the average length of a random arc on a unit circle.



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The moments of random walks

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$$\begin{aligned} W_3(1) &\approx 1.57459723755189365749 \\ W_4(1) &\approx 1.79909248 \\ W_5(1) &\approx 2.00816 \end{aligned}$$

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- Hard to evaluate numerically to high precision.
  Monte-Carlo integration gives approximations with an asymptotic error of $O(1/\sqrt{N})$ where N is the number of sample points.

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2	1.273	2.000	3.395	6.000	10.87	20.00	37.25
3	1.575	3.000	6.452	15.00	36.71	93.00	241.5
4	1.799	4.000	10.12	28.00	82.65	256.0	822.3
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The even moments

n	$s=0$	$s=2$	$s=4$	$s=6$	$s=8$	$s=10$	Sloane's
2	1	2	6	20	70	252	A000984
3	1	3	15	93	639	4653	A002893
4	1	4	28	256	2716	31504	A002895
5	1	5	45	545	7885	127905	A169714
6	1	6	66	996	18306	384156	A169715

EG

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$ **Apéry-like**

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}$$ **Domb numbers**

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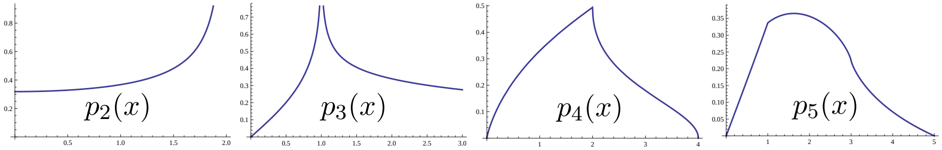
Domb numbers

THM Borwein-Nuyens-S-Wan, 2010

$$W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

Densities of random walks



$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3} \right)$$

classical
with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re}\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6} \end{matrix} \middle| \frac{\left(16-x^2\right)^3}{108x^4} \right)$$

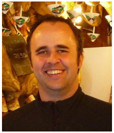
new
BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4} \Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$

Based on joint work with:



Mathew Rogers
(University of Montreal)

Ramanujan's series for $1/\pi$

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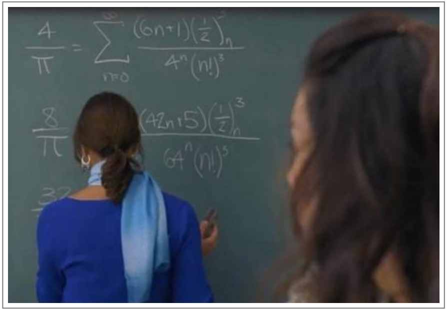
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \ldots$$

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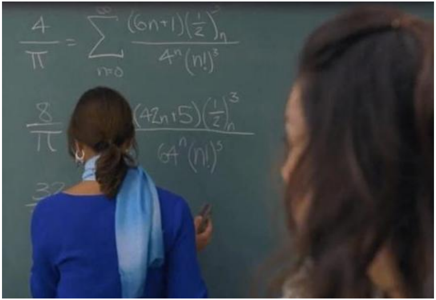
$$\frac{\mathbf{16}}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

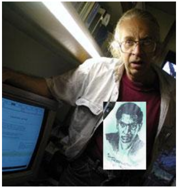


- Starred in High School Musical, a 2006 Disney production

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

Another one of Ramanujan's series

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103 + 26390n}{396^{4n}}$$



- Used by R. W. Gosper in 1985 to compute $17,526,100$ digits of π

  Correctness of first 3 million digits showed that the series sums to $1/\pi$ in the first place.

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- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG
Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

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- Sun offered a $520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

A brief guide to proving series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n + 5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \frac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Positivity of rational functions

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}$$

Based on joint work with:

Wadim Zudilin
(University of Newcastle)

Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

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$$S(x,y,z)=\frac{1}{1-(x+y+z)+\frac{3}{4}(xy+yz+zx)}$$

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n}\binom{n}{k}^3$.

Positivity of rational functions

CONJ Kauers–Zeilberger 2008

The following rational function is positive:

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- Would imply conjectured positivity of Lewy–Askey rational function

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Recent proof of non-negativity by Scott and Sokal, 2013

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PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

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THM S-Zudilin 2013

$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2xy} \quad \text{is positive}$$

$\iff$ diagonal of F and $F|_{x_d=0}$ are positive

Summary and some open problems

- Apéry-like numbers are integer solutions to certain three-term recurrences
  - is the experimental list complete?
  - higher-order analogs, Calabi–Yau DEs
- Apéry-like numbers have interesting properties
  - modular parametrization; uniform explanation?
  - supercongruences; still open in several cases
- Apéry-like numbers occur in interesting places
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties
  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990