Multivariate Apéry numbers and supercongruences of rational functions

Recent Developments in Number Theory
AMS Spring Central Sectional Meeting, Lubbock

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

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satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

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- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions

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- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

Apéry-like numbers

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^{12n}}_{\text{modular function}}.$$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

- The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}.$$

Chowla–Cowles–Cowles '80
Gessel '82
Beukers, Coster '85, '88

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EG Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \mod p^2.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \mod p^3.$$

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \mod p^{3r}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	open!!
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n}\right)$	Osburn–Sahu–S '13
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k}\binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '13
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l}\binom{k}{l}\binom{k+l}{n}$	open
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88

Sources for (non-super) congruences

$$a(np^r) \equiv a(np^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- $a(n)$ is **realizable** if there is some map $T : X \to X$ such that

$$a(n) = \#\{x \in X : T^n x = x\}. \qquad \text{“points of period } n\text{”}$$

In that case, (C) holds. Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, (C) characterizes realizability.

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- Let $\Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be a Laurent polynomial.
  If the Newton polyhedron of Λ contains the origin as its only interior point, then $a(n) = \operatorname{ct} \Lambda(x)^n$ satisfies (C). van Straten–Samol '09

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

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The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

Apéry numbers as diagonals

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- Previously known: they are also the diagonal of (Christol, '84)

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to prove, but much harder to discover.

Apéry numbers as diagonals

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- Univariate generating function:

$$\sum_{n \geqslant 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \right| - \frac{1024x}{(1 - x + z)^4} \right),$$

where $z = \sqrt{1 - 34x + x^2}$.

Apéry numbers as diagonals

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THM **S 2013** The Apéry numbers are the diagonal coefficients of

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically (pun intended) leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \mod 8, & \text{if } n \text{ even}, \\ 5 \mod 8, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

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Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$. For primes $p \geqslant 5$,

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \mod p^{3r}.$$

Multivariable supercongruences

- Denote with $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ the coefficients of

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THM S 2013 Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$. For primes $p \geqslant 5$,

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- Note that if
$$\sum_{n \geqslant 0} a(n) x^n = F(x), \qquad \zeta_p = e^{2\pi i/p}$$
then
$$\sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x).$$

- Hence, both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.

Multivariable supercongruences

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THM S 2013 Let $\boldsymbol{n} = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$. For primes $p \geqslant 5$,

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- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_3}{k} \binom{n_1 + n_2 - k}{n_1} \binom{n_3 + n_4 - k}{n_3}.$$

Multivariable supercongruences

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- Because $A(n-1) = A(-n, -n, -n, -n)$, we also have

$$A(mp^r - 1) \equiv A(mp^{r-1} - 1) \mod p^{3r}.$$

Beukers '85

- The Franel numbers

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$$

are the diagonal coefficients of both

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3) - x_1 x_2 x_3}, \quad \frac{1}{1-(x_1+x_2+x_3)+4x_1 x_2 x_3}.$$

Further examples

- The Franel numbers

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- The multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \mod p^{3r}$$

appear to hold in both cases. Open in the second case.

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof for all of them
  - uniform explanation
  - multivariable extensions
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - reason for modularity
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - q-analogs
  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
Preprint, 2013

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
Preprint, 2013

Fuller version of main result

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Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}_{>0}^\ell$ with $d = \lambda_1 + \ldots + \lambda_\ell$, and set $s(j) = \lambda_1 + \ldots + \lambda_{j-1}$. Define $A_\lambda(\boldsymbol{n})$ by

$$\left(\prod_{j=1}^{\ell} \left[1 - \sum_{r=1}^{\lambda_j} x_{s(j)+r} \right] - x_1 x_2 \cdots x_d \right)^{-1} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$ and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$