Trigonometric Dirichlet series and Eichler integrals

Number Theory and Experimental Mathematics Day
Dalhousie University

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Based on joint work with:

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Secant zeta function

- Lalín, Rodrigue and Rogers introduce and study

$$\psi_s(\tau) = \sum_{n=1}^{\infty} \frac{\sec(\pi n \tau)}{n^s}.$$

- Clearly, $\psi_s(0) = \zeta(s)$. In particular, $\psi_2(0) = \frac{\pi^2}{6}$.

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LRR '13

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CONJ LRR '13 For positive integers m, r,

$$\psi_{2m}(\sqrt{r}) \in \mathbb{Q} \cdot \pi^{2m}.$$

Basic examples of trigonometric Dirichlet series

- Euler's identity:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -\frac{1}{2}(2\pi i)^{2m} \frac{B_{2m}}{(2m)!}$$

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- Half of the Clausen and Glaisher functions reduce, e.g.,

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n \tau)}{n^{2m}} = \mathrm{poly}_m(\tau), \qquad \mathrm{poly}_1(\tau) = \frac{\pi^2}{12}\left(3\tau^2 - 6\tau + 2\right).$$

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- Ramanujan investigated trigonometric Dirichlet series of similar type. From his first letter to Hardy:

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^7} = \frac{19\pi^7}{56700}$$

In fact, this was already included in a general formula by Lerch.

Secant zeta function: Convergence

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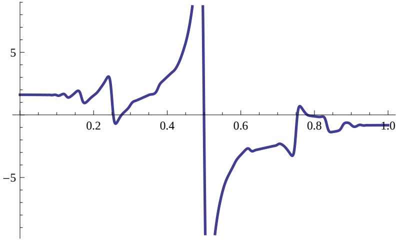
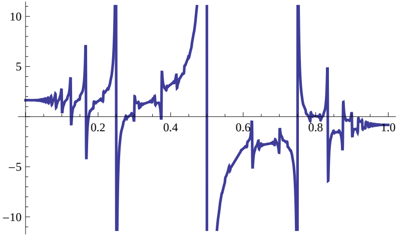


$\mathrm{Re}\ \psi_2(\tau + \varepsilon i)$ with $\varepsilon = 1/100$



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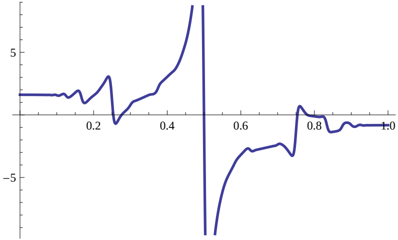

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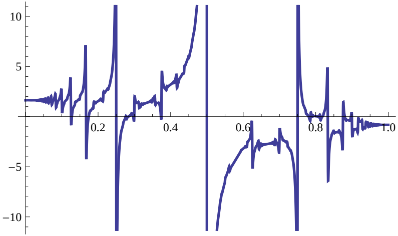

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The series $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ converges absolutely if

1. $\tau = p/q$ with q odd and $s > 1$,
2. τ is algebraic irrational and $s \geqslant 2$.

- Proof uses Thue–Siegel–Roth, as well as a result of Worley when $s = 2$ and τ is irrational

Secant zeta function: Functional equation

- Obviously, $\psi_s(\tau) = \sum \frac{\sec(\pi n \tau)}{n^s}$ satisfies $\psi_s(\tau + 2) = \psi_s(\tau)$.

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$$(1+\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1+\tau}\right) - (1-\tau)^{2m-1}\psi_{2m}\left(\frac{\tau}{1-\tau}\right)$$
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proof Collect residues of the integral

$$I_C = \frac{1}{2\pi i}\int_C \frac{\sin(\pi\tau z)}{\sin(\pi(1+\tau)z)\sin(\pi(1-\tau)z)}\frac{\mathrm{d}z}{z^{s+1}}.$$

C are appropriate circles around the origin such that $I_C \to 0$ as $\mathrm{radius}(C) \to \infty$. □

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- Hence, ψ_{2m} transforms under $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $R^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$,
- Together, with $-I$, these two matrices generate $\Gamma(2)$.

Secant zeta function: Special values

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- Hence, $\psi_2(\sqrt{2}) = -\dfrac{\pi^2}{3}$.

Modular forms

There's a saying attributed to Eichler that there are five fundamental operations of arithmetic: addition, subtraction, multiplication, division, and modular forms.

Andrew Wiles (BBC Interview, "The Proof", 1997)

DEF Actions of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$:

- on $\tau \in \mathcal{H}$ by $$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d},$$
- on $f : \mathcal{H} \to \mathbb{C}$ by $$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma \cdot \tau).$$

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Eisenstein series of weight $2k$:

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$$\cot(\pi\tau) = \frac{1}{\pi} \sum_{j\in\mathbb{Z}} \frac{1}{\tau+j}$$

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- In other words, $\sum \frac{\cot(\pi n\tau)}{n^{2k-1}}$ is an **Eichler integral** of G_{2k}.

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- F is an **Eichler integral** if $D^{k-1}F$ is modular of weight k. $D = q\frac{\mathrm{d}}{\mathrm{d}q}$

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- $\mathrm{poly}(\tau)$ is a **period polynomial** of the modular form f.
  The period polynomial encodes the critical L-values of f.
- For a modular form $f(\tau) = \sum a(n) q^n$ of weight k, define

$$\tilde{f}(\tau) = \frac{(-1)^k \Gamma(k-1)}{(2\pi i)^{k-1}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{k-1}} q^n.$$

If $a(0) = 0$, $\tilde{f}$ is an Eichler integral as defined above.

Ramanujan already knew all that

THM Ramanujan, Grosswald

For $\alpha, \beta > 0$ such that $\alpha\beta = \pi^2$ and $m \in \mathbb{Z}$,

$$\alpha^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\alpha n} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{\zeta(2m+1)}{2} + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\beta n} - 1} \right\}$$

$$-2^{2m} \sum_{n=0}^{m+1} (-1)^n \frac{B_{2n}}{(2n)!} \frac{B_{2m-2n+2}}{(2m-2n+2)!} \alpha^{m-n+1} \beta^n.$$

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- In terms of $\xi_s(\tau) = \sum \frac{\cot(\pi n \tau)}{n^s}$, Ramanujan's formula becomes

$$\xi_{2k-1}|_{2-2k}(S-1) = (-1)^k (2\pi)^{2k-1} \sum_{s=0}^{k} \frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}\tau^{2s-1}.$$

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- Equivalently, the period "polynomial" of the Eisenstein series G_{2k} is

$$\tilde{G}_{2k}|_{2-2k}(S-1) = \frac{(2\pi i)^{2k}}{2k-1}\left[\sum_{s=0}^{k}\frac{B_{2s}}{(2s)!}\frac{B_{2k-2s}}{(2k-2s)!}X^{2s-1} + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2}-1)\right].$$

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- $\sum_{m,n\in\mathbb{Z}}' \frac{\chi_{-4}(n)}{(m\tau+n)^{2k+1}}$ is an Eisenstein series of weight $2k+1$.
- More generally, we have the Eisenstein series

$$E_k(\tau;\chi,\psi) = \sum_{m,n\in\mathbb{Z}}' \frac{\chi(m)\psi(n)}{(m\tau+n)^k},$$

  where χ and ψ are Dirichlet characters modulo L and M.
- We assume $\chi(-1)\psi(-1) = (-1)^k$. Otherwise, $E_k(\tau;\chi,\psi) = 0$.

THM Berndt-S 2013

For $k \geqslant 3$, primitive χ, $\psi \neq 1$, and n such that $L|n$,

$R^n = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$

$$\tilde{E}_k(X;\chi,\psi)|_{2-k}(1-R^n)$$
$$= \text{const} \sum_{s=0}^{k} \frac{B_{k-s,\bar{\chi}}}{(k-s)!L^{k-s}} \frac{B_{s,\bar{\psi}}}{s!M^s} X^{s-1}|_{2-k}(1-R^n).$$

$$\text{const} = -\chi(-1)G(\chi)G(\psi)\frac{(2\pi i)^k}{k-1}$$

- The **generalized Bernoulli numbers** appear because

$$L(1-n,\chi) = -B_{n,\chi}/n.$$

($n > 0$, primitive χ with $\chi(-1) = (-1)^n$)

- Note that $X^{s-1}|_{2-k}(1-R^n) = X^{s-1}(1-(nX+1)^{k-1-s})$.

Application: Grosswald-type formula for Dirichlet L-values

THM
Berndt-S
2013

For $\alpha \in \mathcal{H}$, such that $R_k(\alpha; \bar{\chi}, 1) = 0$ and $\alpha^{k-2} \neq 1$,
($k \geqslant 3$, χ primitive, $\chi(-1) = (-1)^k$)

$$L(k-1, \chi) = \frac{k-1}{2\pi i(1-\alpha^{k-2})} \left[\tilde{E}_k \left(\frac{\alpha - 1}{L}; \chi, 1 \right) - \alpha^{k-2} \tilde{E}_k \left(\frac{1 - 1/\alpha}{L}; \chi, 1 \right) \right]$$
$$= \frac{2}{1-\alpha^{k-2}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{k-1}} \left[\frac{1}{1 - e^{2\pi i n(1-\alpha)/L}} - \frac{\alpha^{k-2}}{1 - e^{2\pi i n(1/\alpha - 1)/L}} \right].$$

THM Berndt-S 2013

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THM Gun–Murty–Rath 2011

As $\beta \in \mathcal{H}$, $\beta^{2k-2} \neq 1$, ranges over algebraic numbers, the values

$$\frac{1}{\pi} \left[\tilde{E}_{2k}(\beta; 1, 1) - \beta^{2k-2} \tilde{E}_{2k}(-1/\beta; 1, 1) \right]$$

contain at most one algebraic number.

Unimodular polynomials

DEF $p(x)$ is **unimodular** if all its zeros have absolute value 1.

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EG **Lehmer**

$$x^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

has only the two real roots 0.850, 1.176 off the unit circle.

Lehmer's conjecture: $1.176\ldots$ is the smallest Mahler measure (greater than 1)

Ramanujan polynomials

- Following Gun–Murty–Rath, the **Ramanujan polynomials** are

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

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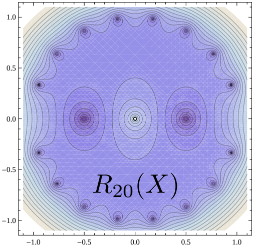
THM Murty-Smyth-Wang '11: All nonreal zeros of $R_k(X)$ lie on the unit circle.
For $k \geqslant 2$, $R_{2k}(X)$ has exactly four real roots which approach $\pm 2^{\pm 1}$.



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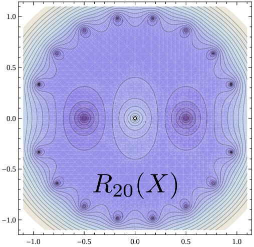
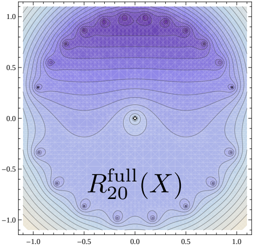
THM Lalín-Smyth '13: $R_{2k}(X) + \frac{\zeta(2k-1)}{(2\pi i)^{2k-1}}(X^{2k-2} - 1)$ is unimodular.





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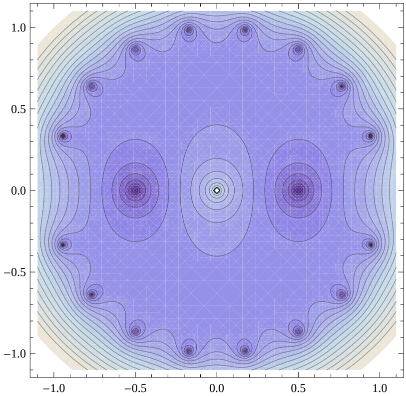
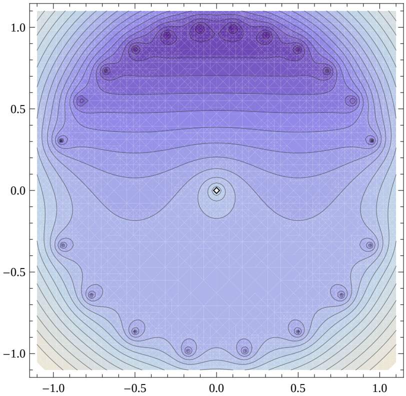


$R_{20}(X)$



$R_{20}^{\text{full}}(X)$

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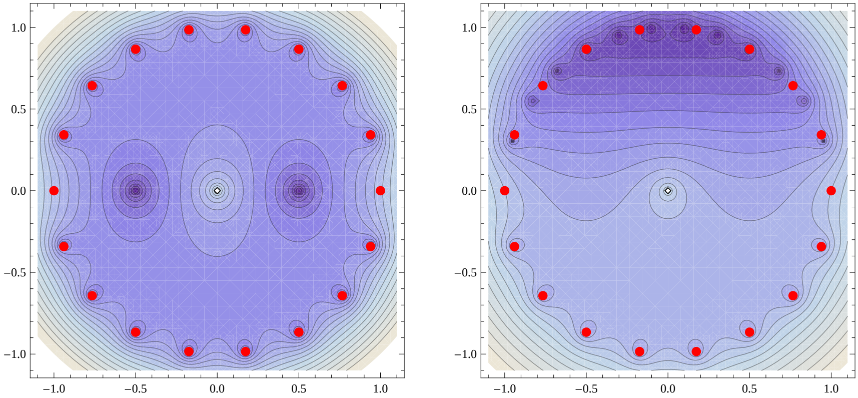

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Unimodularity of period polynomials

THM Conrey-Farmer-Imamoglu 2012

For any Hecke cusp form (for $\mathrm{SL}_2(\mathbb{Z})$), the odd part of its period polynomial has

- trivial zeros at 0, ± 2, $\pm \frac{1}{2}$,
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THM El-Guindy–Raji 2013

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THM El-Guindy–Raji 2013

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Q What about higher level?

- Consider the following **generalized Ramanujan polynomials**:

$$R_k(X;\chi,\psi) = \sum_{s=0}^{k} \frac{B_{s,\chi}}{s!} \frac{B_{k-s,\psi}}{(k-s)!} \left(\frac{X-1}{M}\right)^{k-s-1} (1-X^{s-1})$$

- Essentially, period polynomials: χ, ψ primitive, nonprincipal

$$R_k(LX+1;\chi,\psi) = \text{const} \cdot \tilde{E}_k(X;\bar{\chi},\bar{\psi})|_{2-k}(1-R^L)$$

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PROP
Berndt-S
2013

- For even $k>1$,

$$R_k(X;1,1) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}.$$

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PROP Berndt-S 2013

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CONJ If χ,ψ are nonprincipal real, then $R_k(X;\chi,\psi)$ is unimodular.

Generalized Ramanujan polynomials

EG

$$R_k(X; \chi, 1)$$

For χ real, conjecturally unimodular unless:

- $\chi = 1$: $R_{2k}(X; 1, 1)$ has real roots approaching $\pm 2^{\pm 1}$
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EG

$$R_k(X; 1, \psi)$$

Conjecturally:

- unimodular for ψ one of
  $$3-, 4-, 5+, 8\pm, 11-, 12+, 13+, 19-, 21+, 24+, \ldots$$
- all nonreal roots on the unit circle if ψ is one of
  $$1+, 7-, 15-, 17+, 20-, 23-, 24-, \ldots$$
- four nonreal zeros off the unit circle if ψ is one of
  $$35-, 59-, 83-, 131-, 155-, 179-, \ldots$$

Generalized Ramanujan polynomials

- A second kind of **generalized Ramanujan polynomials**:

$$R_k(X) = \sum_{s=0}^{k} \frac{B_s}{s!} \frac{B_{k-s}}{(k-s)!} X^{s-1}$$

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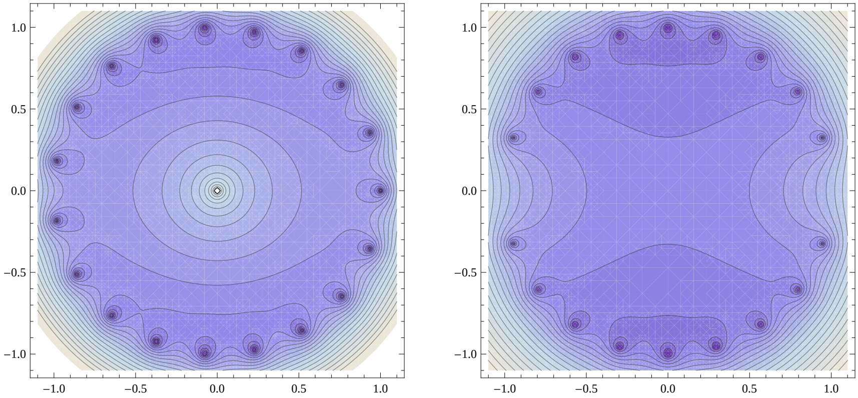
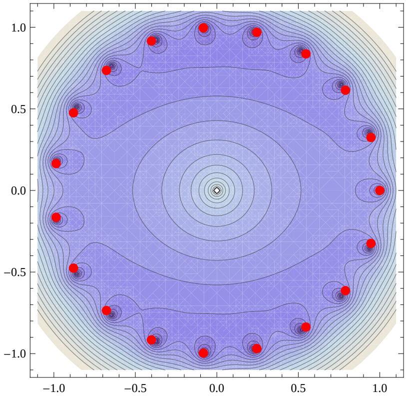
CONJ If χ is nonprincipal real, then $S_k(X;\chi,\chi)$ is unimodular (up to trivial zero roots).

Generalized Ramanujan polynomials



$R_{19}(X; 1, \chi_{-4})$ $S_{20}(X; \chi_{-4}, \chi_{-4})$

Generalized Ramanujan polynomials



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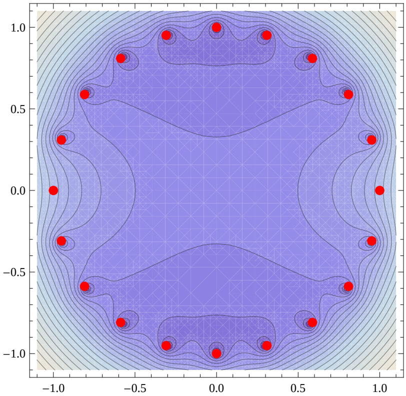

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Special values of trigonometric Dirichlet series

EG Ramanujan

$$\sum_{n=0}^{\infty} \frac{\tanh((2n+1)\pi/2)}{(2n+1)^3} = \frac{\pi^3}{32}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\operatorname{csch}(\pi n)}{n^3} = \frac{\pi^3}{360}$$

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EG Berndt 1976-78

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EG Komori-Matsumoto-Tsumura 2013

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THM S 2014

Let $r \in \mathbb{Q}$, and let $a, b, s \in \mathbb{Z}$ be such that $s \geqslant \max(a, b, 1) + 1$, s and b have the same parity, and $a + b \geqslant 0$. Then,

$$\sum_{n=1}^{\infty} \frac{\operatorname{trig}^{a,b}(\pi n\sqrt{r})}{n^s} \in (\pi\sqrt{r})^s\mathbb{Q}, \qquad \operatorname{trig}^{a,b} = \sec^a \csc^b.$$

EG
S 2014

$$\sum_{n=1}^{\infty} \frac{\sec^2(\pi n\sqrt{5})}{n^4} = \frac{14}{135}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\cot^2(\pi n\sqrt{5})}{n^4} = \frac{13}{945}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\csc^2(\pi n\sqrt{11})}{n^4} = \frac{8}{385}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\sec^3(\pi n\sqrt{2})}{n^4} = -\frac{2483}{5220}\pi^4$$

$$\sum_{n=1}^{\infty} \frac{\tan^3(\pi n\sqrt{6})}{n^5} = \frac{35,159}{17,820\sqrt{6}}\pi^4$$

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

B. Berndt, A. Straub
On a secant Dirichlet series and Eichler integrals of Eisenstein series
Preprint, 2013

A. Straub
Special values of trigonometric Dirichlet series and Eichler integrals
In preparation, 2014