Supercongruences for Apéry-like numbers

$npqr^2$ **seminar**
NIE, Singapore

Armin Straub

August 13, 2014

University of Illinois at Urbana–Champaign

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

Jon Borwein
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Dirk Nuyens
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James Wan
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Wadim Zudilin
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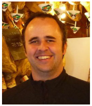

Mathew Rogers
U. of Montreal, CA

Apéry numbers and the irrationality of $\zeta(3)$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q
Beukers, Zagier

Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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Q (Beukers, Zagier): Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions
  - 6 sporadic solutions
- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

Apéry-like numbers

- Hypergeometric and Legendrian solutions have generating functions

$$ {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\right|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\right|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$.

- The six sporadic solutions are:

(a,b,c)	$A(n)$
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k}\binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$
$(12,4,16)$	$\sum_k \binom{n}{k}^2\binom{2k}{n}^2$
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$

Modularity of Apéry-like numbers

- The **Apéry numbers** $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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EG

Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

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- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n}+\binom{4n-5k}{3n}\right)$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$	Chan–Cooper–Sica '10 Osburn–Sahu '11
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$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2\binom{n}{l}\binom{k}{l}\binom{k+l}{n}$	open
$(17,5,1)$	$\sum_k \binom{n}{k}^2\binom{n+k}{n}^2$	Beukers, Coster '87-'88

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{"points of period } n\text{"}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \operatorname{ct} \Lambda(x)^n$ van Straten–Samol '09

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

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CONJ Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

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THM Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

Applications of Apéry-like numbers

- Random walks

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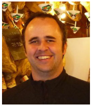
Dirk Nuyens
K.U.Leuven, BE

James Wan
SUTD, SG

Wadim Zudilin
U. of Newcastle, AU

- Series for $1/\pi$



Mat Rogers
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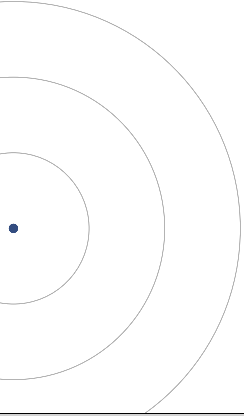
- Positivity of rational functions

Wadim Zudilin
U. of Newcastle, AU

Example I: Random walks

n steps in the plane
(length 1, random direction)



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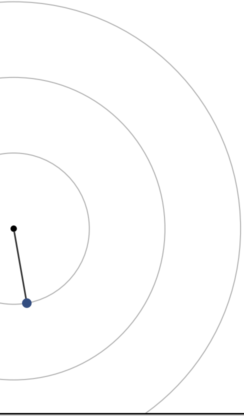

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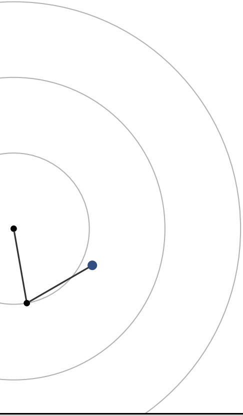

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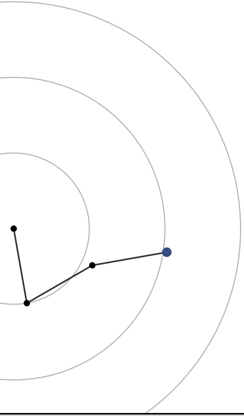

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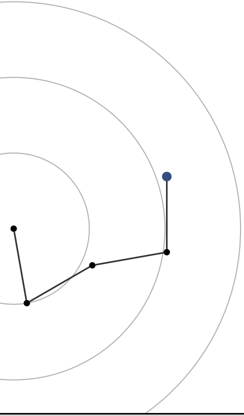

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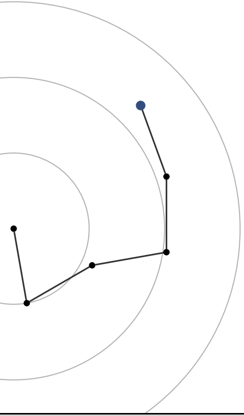

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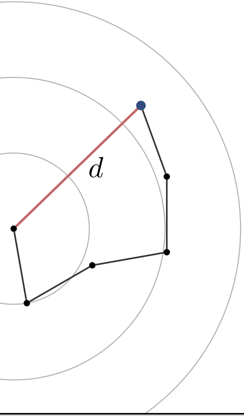

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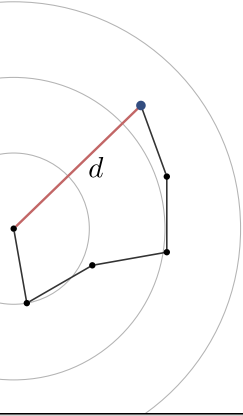
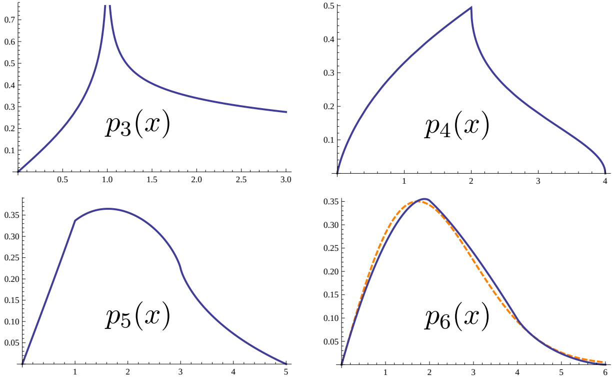


- $p_n(x)$ — probability density of distance traveled



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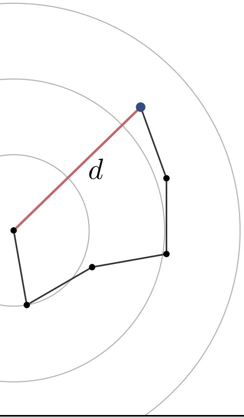

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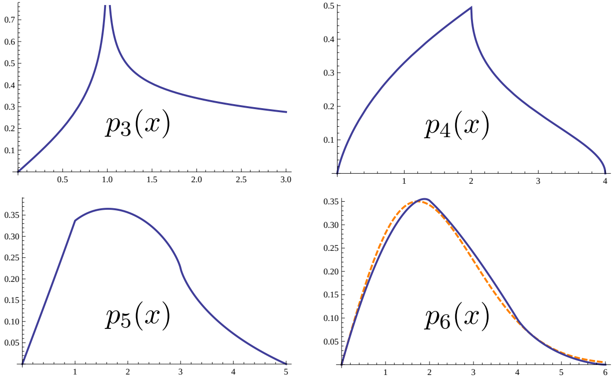

- $W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$ — probability moments

$$W_2(1) = \frac{4}{\pi}, \qquad W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

classical — Borwein–Nuyens–S–Wan, 2010

Example I: Random walks

- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

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THM Borwein-Nuyens-S-Wan 2010

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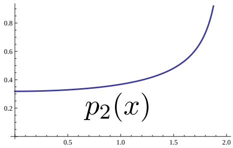

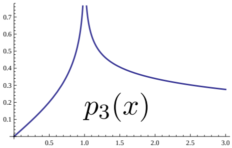

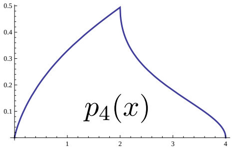

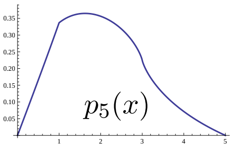

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$ easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3}\\ 1\end{matrix}\middle| \frac{x^2(9-x^2)^2}{(3+x^2)^3}\right)$$ classical with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re}\, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{5}{6}, \frac{7}{6}\end{matrix}\middle| \frac{(16-x^2)^3}{108x^4}\right)$$ new BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Example II: Series for $1/\pi$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103+26390n}{396^{4n}}$$

- Last series used by Gosper in 1985 to compute $17,526,100$ digits of π
- First proof of all of Ramanujan's 17 series by Borwein brothers

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

Example II: Series for $1/\pi$

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

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EG Chan-Chan-Liu 2003

For the Domb numbers $D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} D(n) \frac{5n+1}{2^{6n}}.$$

- Sun offered a \$520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

Example II: Series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \dfrac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Example III: Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$.

Example III: Positivity of rational functions

CONJ Kauers-Zeilberger 2008

The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

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Recent proof of non-negativity by Scott and Sokal, 2013

PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Example III: Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG $\frac{1}{1+x+y}$ has positive diagonal coefficients but is not positive.

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Q F positive $\Longleftrightarrow$ diagonal of F and $F|_{x_d=0}$ are positive?

THM S-Zudilin 2013

$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2 xy} \quad \text{is positive}$$

$\Longleftrightarrow$ diagonal of F and $F|_{x_d=0}$ are positive

Multivariate Apéry numbers

Apéry numbers as diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1,\ldots,n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F \in K[[x_1, \ldots, x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1, \ldots, x_d)$.

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EG Christol 1984

The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1)\left[(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5) - x_1 x_2 x_3\right]}.$$

- Such identities are routine to prove, but much harder to discover.

Apéry numbers as diagonals

THM
S 2013

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THM
S 2013

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THM
MacMahon 1915

For $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geqslant 0}^n$,

$$[\boldsymbol{x}^{\boldsymbol{m}}] \frac{1}{\det(I_n - BX)} = [\boldsymbol{x}^{\boldsymbol{m}}] \prod_{i=1}^{n} \left(\sum_{j=1}^{n} B_{i,j} x_j \right)^{m_i},$$

where $B \in \mathbb{C}^{n \times n}$ and X is the diagonal matrix with entries $x_1, \ldots, x_n$.

Apéry numbers as diagonals

THM S 2013

The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

THM MacMahon 1915

For $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{m}=(m_1,\ldots,m_n)\in\mathbb{Z}_{\geqslant 0}^n$,

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where $B\in\mathbb{C}^{n\times n}$ and X is the diagonal matrix with entries $x_1,\ldots,x_n$.

$$B=\begin{pmatrix}1&1&1&0\\1&1&0&0\\0&0&1&1\\0&1&1&1\end{pmatrix},\quad X=\begin{pmatrix}x_1&&&\\&x_2&&\\&&x_3&\\&&&x_4\end{pmatrix}$$

$$A(\boldsymbol{n})=[\boldsymbol{x}^{\boldsymbol{n}}](x_1+x_2+x_3)^{n_1}(x_1+x_2)^{n_2}(x_3+x_4)^{n_3}(x_2+x_3+x_4)^{n_4}$$

Apéry numbers as diagonals

THM
S 2013

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- The coefficients are the multivariate Apéry numbers

$$A(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}.$$

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

Apéry numbers as diagonals

THM S 2013 The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}^4_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even}, \\ 5 \pmod{8}, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

Multivariate supercongruences

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S 2013

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

Multivariate supercongruences

THM S 2013

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- $\sum_{n \geqslant 0} a(n) x^n = F(x) \quad \Longrightarrow \quad \sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x) \qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

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- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}.$$

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- Because $A(n-1) = A(-n,-n,-n,-n)$, we also find

$$A(mp^r-1) \equiv A(mp^{r-1}-1) \pmod{p^{3r}}.$$

Beukers '85

Many more conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

  $1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$

$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$

$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$

$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$

$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$

$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

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$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$
$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$
$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$
$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$
$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

CONJ S 2014 The coefficients $B(\boldsymbol{n})$ of each of these satisfy, for $p \geqslant 5$,

$$B(\boldsymbol{n}p^r) \equiv B(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

A simple conjectural tip of an iceberg

CONJ S 2013 The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Franel numbers**

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

A simple conjectural tip of an iceberg

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- Here, the diagonal coefficients are the **Franel numbers**

$$F(n) = \sum_{k=0}^{n} \binom{n}{k}^3.$$

- The Franel numbers also appear as the diagonal coefficients of

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)-x_1x_2x_3},$$

for which we can prove the above multivariate supercongruences.

Extended version of main result

THM S 2014

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}_{>0}^\ell$ with $d = \lambda_1 + \ldots + \lambda_\ell$, and set $s(j) = \lambda_1 + \ldots + \lambda_{j-1}$. Define $A_\lambda(\boldsymbol{n})$ by

$$\left(\prod_{j=1}^{\ell} \left[1 - \sum_{r=1}^{\lambda_j} x_{s(j)+r} \right] - x_1 x_2 \cdots x_d \right)^{-1} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$ and integers $r \geqslant 1$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

Further examples

EG The Apéry-like numbers, associated with $\zeta(2)$,

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

are the diagonal coefficients of the rational function

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^3} B(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

COR We find

$$B(\boldsymbol{n}) = \sum_{k \in \mathbb{Z}} \binom{n_1}{k} \binom{n_1+n_2-k}{n_1} \binom{n_3}{k},$$

and, for primes $p \geqslant 5$,

$$B(p^r \boldsymbol{n}) \equiv B(p^{r-1} \boldsymbol{n}) \pmod{p^{3r}}.$$

- The diagonal case recovers supercongruences of Coster, 1988.

Further examples

EG The numbers

$d = 3$: Franel, $d = 4$: Yang–Zudilin

$$Y_d(n) = \sum_{k=0}^{n} \binom{n}{k}^d,$$

are the diagonal coefficients of the rational function

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d) - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} Y_d(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

COR We find

$$Y_d(\boldsymbol{n}) = \sum_{k \geqslant 0} \binom{n_1}{k}\binom{n_2}{k} \cdots \binom{n_d}{k},$$

and, for $d \geqslant 2$ and primes $p \geqslant 5$,

$$Y_d(p^r \boldsymbol{n}) \equiv Y_d(p^{r-1} \boldsymbol{n}) \pmod{p^{3r}}.$$

- This generalizes a result of Chan–Cooper–Sica, 2010.

q-analogs

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

- The q-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} = \binom{n}{n-k}_q$$

D1

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$
$$= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q=1$
  and hence invisible in the classical world

The coefficients of q-binomial coefficients

- Here's some q-binomials in **expanded** form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q = q^{18} &+ q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ &+ 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ &+ 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are **unimodal**. Sylvester, 1878

q-binomials: Pascal's triangle

The q-binomials can be build from the q-Pascal rule:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D2

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D2

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1+q & & 1 & \\ 1 & & 1+q(1+q) & & (1+q)+q^2 & & 1 \\ & & & \vdots & & & \end{array}$$

EG

$$\binom{4}{2}_q = 1+q+q^2+q^2(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

q-binomials: combinatorial

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

D3

$w(S) =$ "normalized sum of S"

EG

$$\underbrace{\{1,2\}}_{\to 0}, \underbrace{\{1,3\}}_{\to 1}, \underbrace{\{1,4\}}_{\to 2}, \underbrace{\{2,3\}}_{\to 2}, \underbrace{\{2,4\}}_{\to 3}, \underbrace{\{3,4\}}_{\to 4}$$

$$\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$$

$$\binom{n}{k}_q = \sum_{S \in \binom{n}{k}} q^{w(S)} \quad \text{where } w(S) = \sum_j s_j - j$$

$w(S) =$ "normalized sum of S"

D3

EG

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The coefficient of q^m in $\binom{n}{k}_q$ counts the number of

- k-element subsets of n whose normalized sum is m
- partitions λ of m whose Ferrer's diagram fits in a $k \times (n-k)$ box

q-Chu-Vandermonde

Different representations make different properties apparent!

- Chu-Vandermonde: $\binom{m+n}{k} = \sum_j \binom{m}{j}\binom{n}{k-j}$

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q-binomials: algebraic

Let q be a prime power.

$$\binom{n}{k}_q = \text{number of } k\text{-dim. subspaces of } \mathbb{F}_q^n$$

D4

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- Number of ways to choose k linearly independent vectors in $\mathbb{F}_q^n$:

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

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- Hence the number of k-dim. subspaces of $\mathbb{F}_q^n$ is:

$$\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})} = \binom{n}{k}_q$$

q-binomials: noncommuting variables

Suppose $yx = qxy$ where q commutes with x, y. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

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D5

EG

$$\binom{4}{2}_q x^2 y^2 = xxyy + xyxy + xyyx + yxxy + yxyx + yyxx$$
$$= (1 + q + q^2 + q^2 + q^3 + q^4) x^2 y^2$$

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$$= (1 + q + q^2 + q^2 + q^3 + q^4)x^2y^2$$

- Let $X \cdot f(x) = xf(x)$ and $Q \cdot f(x) = f(qx)$. Then:

$$QX \cdot f(x) = qxf(qx) = qXQ \cdot f(x)$$

Summary: the q-binomial coefficient

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

- Via a q-version of **Pascal's rule**
- **Combinatorially**, as the generating function of the element sums of k-subsets of an n-set
- **Algebraically**, as the number of k-dimensional subspaces of $\mathbb{F}_q^n$
- Via a **binomial theorem** for noncommuting variables
- Not touched here:
  - **analytical** definition via q-integral representations
  - **quantum groups** arising in representation theory and physics

A q-analog of Babbage's congruence

- Using q-Chu-Vandermonde
$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$
$$\equiv q^{p^2} + 1 \pmod{[p]_q^2}$$
- Again, $[p]_q$ divides $\binom{p}{k}_q$ unless $k = 0$ or $k = p$.

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THM $\binom{2p}{p}_q \equiv [2]_{q^{p^2}} \pmod{[p]_q^2}$

- Actually, this argument extends to show:

THM Clark 1995 $\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \pmod{[p]_q^2}$

- Using q-Chu-Vandermonde

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$

- Aga

THM

- Act

Similar results by Andrews; e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$

George Andrews
q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher
Discrete Mathematics 204, 1999

THM Clark 1995

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \pmod{[p]_q^2}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM
S 2011

For primes $p \geqslant 5$,

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}.$$

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EG

Choosing $p=13$, $a=2$, and $b=1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}$ is an irreducible polynomial with integer coefficients.

Just coincidence?

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}$$

- Ernst Jacobsthal (1952) proved that Ljunggren's classical congruence holds modulo p^{3+r} where r is the p-adic valuation of

$$ab(a-b)\binom{a}{b} = 2a\binom{a}{b+1}\binom{b+1}{2}.$$

- It would be interesting to see if this generalization has a nice analog in the q-world.

The case of composite numbers

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \pmod{[p]_q^3}$$

- Note that $\frac{n^2-1}{12}$ is an integer if $(n,6)=1$.

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- Note that $\frac{n^2-1}{12}$ is an integer if $(n,6)=1$.
- Ljunggren's q-congruence holds modulo $\Phi_n(q)^3$
  over integer coefficient polynomials if $(n,6)=1$ — otherwise we get rational coefficients.

EG
$n=35$, $a=2$, $b=1$

$$\binom{70}{35}_q = 1 + q^{1225} - 102(q^{35}-1)^2 + \Phi_{35}(q)^3 f(q),$$

where $f(q) = 102 + 307q + 617q^2 + \ldots + q^{1152}$.

Note that Φ_{35} has degree 24.

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EG $n=12$, $a=2$, $b=1$

$$\binom{24}{12}_q = 1 + q^{144} - \frac{143}{12}(q^{12}-1)^2 + \frac{1}{12}(\underbrace{1-q^2+q^4}_{\Phi_{12}(q)})^3 f(q),$$

where $f(q) = 143 + 12q + 453q^2 + \ldots + 12q^{131}$.

THM S 2014

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfy $A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(pn) - A_{q^{p^2}}(n) \equiv -\frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

THM S 2014 The q-analog of the Apéry numbers, defined as

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$$A_q(pn) - A_{q^{p^2}}(n) \equiv -\frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

- The numbers $f(n)$ can be expressed as $\qquad 0, 5, 292, 13005, 528016, \ldots$

$$f(n) = \sum_{k=0}^{n} g(n,k) \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad g(n,k) = k(2n-k) + \frac{k^4}{(n+k)^2}.$$

- Similar congruences for other Apéry-like numbers?

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof for all of them
  - uniform explanation
  - multivariable extensions
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - reason for modularity
  - modular supercongruences

Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - q-analogs
  - ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Preprint, 2014

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

A. Straub
A q*-analog of Ljunggren's binomial congruence*
DMTCS Proceedings: FPSAC 2011, p. 897-902