Congruences for Fishburn numbers modulo prime powers

Partitions, q-series, and modular forms
AMS Joint Mathematics Meetings, San Antonio

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January 11, 2015

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$\xi(3) = 5$

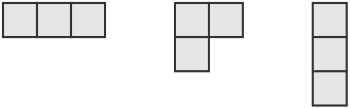
$$\begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Fishburn matrices of size 3

Examples of partitions

- The integer partitions of 3:

$p(3) = 3$



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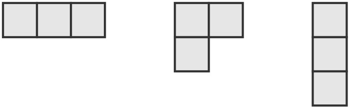

- The Fishburn matrices of size 3:

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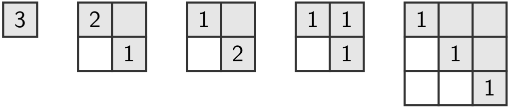


DEF A **Fishburn matrix** is an

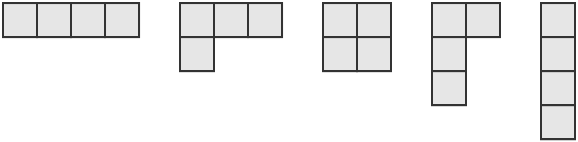
- upper-triangular matrix with entries in $\mathbb{Z}_{\geqslant 0}$, such that
- every row and column contains at least one non-zero entry.

Its **size** is the sum of the entries.

Examples of partitions

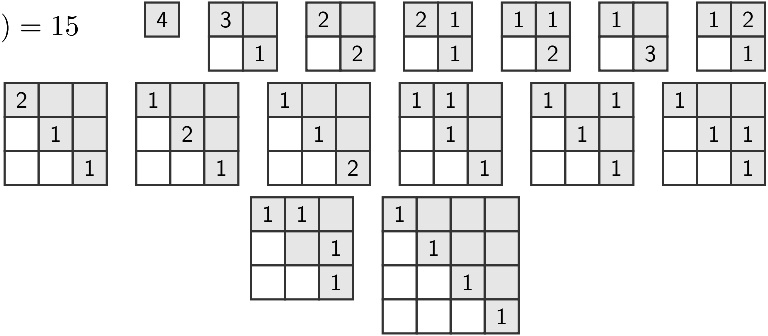
- The integer partitions of 4:

$p(4) = 5$



- The Fishburn matrices of size 4:

$\xi(4) = 15$



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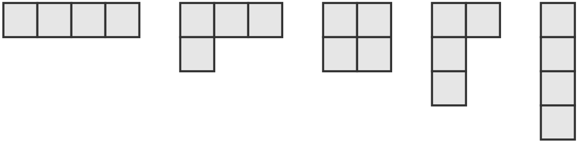

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$$\begin{bmatrix}2 & & \\ & 1 & \\ & & 1\end{bmatrix} \quad \begin{bmatrix}1 & & \\ & 2 & \\ & & 1\end{bmatrix} \quad \begin{bmatrix}1 & & \\ & 1 & \\ & & 2\end{bmatrix} \quad \begin{bmatrix}1 & 1 & \\ & 1 & \\ & & 1\end{bmatrix} \quad \begin{bmatrix}1 & & 1\\ & 1 & \\ & & 1\end{bmatrix} \quad \begin{bmatrix}1 & & \\ & 1 & 1\\ & & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & \\ & & 1\\ & & 1\end{bmatrix} \quad \begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$$

primitive Fishburn matrices

$\xi_{-1}(4) = 5$

Asymptotic facts

$$p(n) : 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots$$

$$p(2015) \approx 7.20 \times 10^{45}$$

$$\xi(n) : 1, 1, 2, 5, 15, 53, 217, 1014, 5335, 31240, 201608, \ldots$$

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THM Hardy–Ramanujan 1918

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$$

THM Zagier 2001

$$\xi(n) \sim \frac{12\sqrt{3n}}{\pi^{5/2}} e^{\pi^2/12} \left(\frac{6}{\pi^2}\right)^n n!$$

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- Primitive Fishburn matrices are those with entries $0, 1$ only.

$$\lim_{n\to\infty} \frac{\#\text{ of primitive Fishburn matrices of size } n}{\#\text{ of Fishburn matrices of size } n \quad (= \xi(n))} = e^{-\pi^2/6} \approx 0.193$$

Jelínek–Drmota, 2011

- The Fishburn numbers $\xi(n)$ have the following generating function.

THM
Zagier 2001

$$\sum_{n \geqslant 0} \xi(n) q^n = \sum_{n \geqslant 0} \prod_{j=1}^{n} (1 - (1-q)^j) = F(1-q)$$

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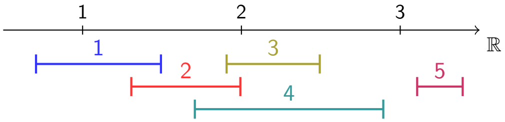
- $F(q)$ is Kontsevich's "strange" function (more later)
- The primitive Fishburn numbers have generating function $F(\frac{1}{1+q})$.
- Garvan introduces the numbers $\xi_{r,s}(n)$ by

$$\sum_{n\geqslant 0} \xi_{r,s}(n) q^n = (1-q)^s \sum_{n\geqslant 0} \prod_{j=1}^{n} (1-(1-q)^{rj}) = (1-q)^s F((1-q)^r).$$

  - $\xi(n) = \xi_{1,0}(n)$
  - $(-1)^n \xi_{-1,0}(n)$ count primitive Fishburn matrices
  - For instance, $\xi_{1,3}(n) = \xi(n) - 3\xi(n-1) + 3\xi(n-2) - \xi(n-3)$.

- An **interval order** is a poset consisting of intervals $I \subseteq \mathbb{R}$ with order given by:

$$I < J \quad \iff \quad i < j \quad \text{for all } i \in I,\ j \in J$$



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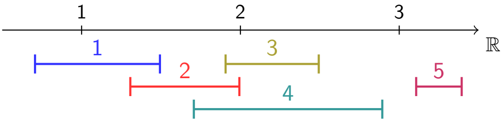

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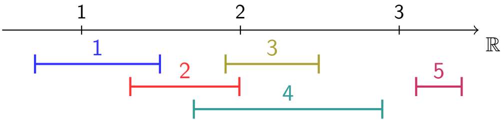
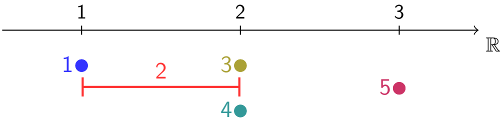


↓ standard representation ↓



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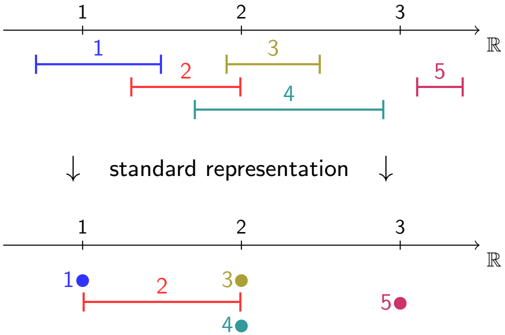

Fishburn matrix M

$$M_{i,j} = \# \text{ of intervals } [i, j]$$

1	1	
	2	
		1

Partition congruences

- Ramanujan proved the striking congruences

$$p(5m-1) \equiv 0 \pmod{5},$$
$$p(7m-2) \equiv 0 \pmod{7},$$
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- Also conjectured generalizations with moduli powers of $5, 7, 11$.

$$p(5^\lambda m - \delta_5(\lambda)) \equiv 0 \pmod{5^\lambda},$$
$$p(7^\lambda m - \delta_7(\lambda)) \equiv 0 \pmod{7^\lambda},$$
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where $\delta_p(\lambda) \equiv -1/24 \pmod{p^\lambda}$.

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$$p(5^\lambda m - \delta_5(\lambda)) \equiv 0 \pmod{5^\lambda},$$
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RK There appear to be additional congruences, such as

$$p(7^2 m - \mathbf{2}, 9, 16, 30) \equiv 0 \pmod{7^2},$$
$$p(5^3 m - 1, \mathbf{26}, 51) \equiv 0 \pmod{5^3}.$$

THM Andrews, Sellers 2014

Let p be a prime, and $j \in \mathbb{Z}_{>0}$ such that

$$\left(\frac{1-24k}{p} \right) = -1 \qquad \text{for } k = 1, 2, \ldots, j. \tag{AS}$$

Then, for all $m \geqslant 1$,

$$\xi(pm - j) \equiv 0 \pmod{p}.$$

EG

$$\begin{aligned} \xi(5m-1) \equiv \xi(5m-2) &\equiv 0 \pmod{5} \\ \xi(7m-1) &\equiv 0 \pmod{7} \\ \xi(11m-1) \equiv \xi(11m-2) \equiv \xi(11m-3) &\equiv 0 \pmod{11} \end{aligned}$$

The Andrews–Sellers congruences

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- Garvan (2014) proved that (AS) may be replaced with

$$\left(\frac{1-24k}{p}\right) \neq 1 \qquad \text{for } k = 1, 2, \ldots, j,$$

and that analogous congruences hold for the generalizations $\xi_{r,s}(n)$.

Garvan's generalized congruences

Andrews–Sellers congruences

$$\xi(5m-1) \equiv \xi(5m-2) \equiv 0 \pmod{5}$$
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Garvan's additional congruences

$$\xi(23m-1) \equiv \xi(23m-2) \equiv \ldots \equiv \xi(23m-5) \equiv 0 \pmod{23}$$

Congruences for primitive Fishburn numbers

$$\xi_{-1}(5m-1) \equiv 0 \pmod{5}$$

Extensions observed by Garthwaite–Rhoades

$$\xi(5m+2) - 2\xi(5m+1) \equiv 0 \pmod{5}$$

Garvan's congruences for r-Fishburn numbers

$$\xi_{23}(7m-1) \equiv \xi_{23}(7m-2) \equiv \xi_{23}(7m-3) \equiv 0 \pmod{7}$$
$$\xi_{-1}(5m-1) - 2\xi_{-1}(5m-2) + \xi_{-1}(5m-3) \equiv 0 \pmod{5}$$

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$$\xi(5^{\boldsymbol{\lambda}}m-1) \equiv \xi(5^{\boldsymbol{\lambda}}m-2) \equiv 0 \pmod{5^{\boldsymbol{\lambda}}}$$

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Garvan's additional congruences

$$\xi(23^{\boldsymbol{\lambda}}m-1) \equiv \xi(23^{\boldsymbol{\lambda}}m-2) \equiv \ldots \equiv \xi(23^{\boldsymbol{\lambda}}m-5) \equiv 0 \pmod{23^{\boldsymbol{\lambda}}}$$

Congruences for primitive Fishburn numbers

$$\xi_{-1}(5m-1) \equiv 0 \pmod{5}$$ ✗

Extensions observed by Garthwaite–Rhoades

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Garvan's congruences for r-Fishburn numbers

$$\xi_{23}(7^{\boldsymbol{\lambda}}m-1) \equiv \xi_{23}(7^{\boldsymbol{\lambda}}m-2) \equiv \xcancel{\xi_{23}(7^{\boldsymbol{\lambda}}m-3)} \equiv 0 \pmod{7^{\boldsymbol{\lambda}}}$$ ✗

$$\xi_{-1}(5^{\boldsymbol{\lambda}}m-1) - 2\xi_{-1}(5^{\boldsymbol{\lambda}}m-2) + \xi_{-1}(5^{\boldsymbol{\lambda}}m-3) \equiv 0 \pmod{5^{\boldsymbol{\lambda}}}$$

- Most, but not all, of these congruences can be lifted to prime powers.

THM
S 2014

Let p be a prime, $p \nmid r$, and $j \in \mathbb{Z}_{>0}$ such that

$$\left(\frac{1 - 24(k+s)/r}{p} \right) = -1 \qquad \text{for } k = 1, 2, \ldots, j. \tag{C}$$

Then, for all $m \geqslant 1$ and λ,

$$\xi_{r,s}(p^\lambda m - j) \equiv 0 \pmod{p^\lambda}.$$

Congruences modulo prime powers

THM S 2014

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In (C), "$= -1$" may be replaced with "$\neq 1$", if $p \geqslant 5$ and

$$\operatorname{digit}_1(s - r/24; p) \neq p - 1. \qquad \text{(*)}$$

- (*) states that $n_1 \neq p - 1$ in $s - r/24 = n_0 + n_1 p + n_2 p^2 + \ldots$

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But (*) is not satisfied: $s - r/24 = 1/24 \equiv -1 \pmod{5^2}$
Indeed, the congruences do not extend to prime powers:

$$\xi_{-1}(5^2 - 1) = 1111583305926812677 0 \equiv 20 \not\equiv 0 \pmod{5^2}$$

Primitive Fishburn matrices

Q Can we give a combinatorial interpretation for any of these congruences?

- In particular, for the number of primitive Fishburn matrices,

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modulo 5 and 7 are explained by Dyson's **rank**.

(rank = largest part of a partition minus the number of its parts)

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- All three congruences are explained by Dyson's speculated **crank**, which was found by Andrews and Garvan (1988).

Kontsevich's "strange" function

DEF

$$F(q) = \sum_{n \geqslant 0} (1-q)(1-q^2) \cdots (1-q^n)$$

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- series terminates when q is a root of unity

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THM Zagier 1999

The following "strange" identity "holds":

$$q^{1/24} F(q) = -\frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) q^{n^2/24}$$

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- LHS agrees at roots of unity with the radial limit of the RHS (and similarly for the derivatives of all orders)
- RHS is the "half-derivative" (up to constants) of the Dedekind eta function

$$\eta(\tau) = q^{1/24} \sum_{n \geqslant 1} (1-q^n) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2/24}.$$

Eichler integrals of half-integral weight

- If $f(\tau) = \sum a(n) q^n$ is a cusp form of integral weight k on $\mathrm{SL}_2(\mathbb{Z})$, then $\tilde{f}(\tau) = \sum n^{1-k} a(n) q^n$ is its **Eichler integral**.

$$\tilde{f}(\tau+1) = \tilde{f}(\tau), \qquad \tau^{k-2}\tilde{f}(-1/\tau) = \tilde{f}(\tau) + \mathrm{poly}(\tau).$$

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EG Eisenstein series

$$\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad \xrightarrow{\text{integrate}} \quad \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k-1}} q^n = \sum_{n=1}^{\infty} \sigma_{1-k}(n) q^n$$

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- $\eta(\tau) = \sum \left(\frac{12}{n}\right) q^{n^2/24}$ has weight $k = 1/2$.
- Its formal Eichler integral is

$$\tilde{\eta}(\tau) = \frac{1}{\sqrt{24}} \sum n \left(\frac{12}{n}\right) q^{n^2/24} = -\frac{1}{\sqrt{6}}\, q^{1/24} F(q).$$

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- $\eta(\tau) = \sum \left(\frac{12}{n}\right) q^{n^2/24}$ has weight $k = 1/2$.
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$$\tilde{\eta}(\tau) = \frac{1}{\sqrt{24}} \sum n \left(\frac{12}{n}\right) q^{n^2/24} = -\frac{1}{\sqrt{6}}\, q^{1/24} F(q).$$

- Bringmann–Rolen (2014) study Eichler integrals of half-integral weight systematically. (inspired by examples of Lawrence–Zagier, 1999)

THM Bringmann Rolen 2014 If $\tilde{f}(\tau)$ is a formal Eichler integral of half-integral weight, then $\tilde{f}(\tau)$ is a quantum modular form.

THM (Bringmann Rolen 2014) If $\tilde{f}(\tau)$ is a formal Eichler integral of half-integral weight, then $\tilde{f}(\tau)$ is a quantum modular form.

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$$f : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{C} \qquad \text{with} \quad f(\tau) - \frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right) = \text{nice}(\tau).$$

Quantum modular forms

THM Bringmann Rolen 2014: If $\tilde{f}(\tau)$ is a formal Eichler integral of half-integral weight, then $\tilde{f}(\tau)$ is a quantum modular form.

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- Above, $\mathrm{nice}(\tau)$ is defined and real analytic on $\mathbb{R}$ (except at one point).

- Is there a general theory of congruences for the coefficients A_n of $\tilde{f}(\tau) = \sum A_n (1-q)^n$ if $\tilde{f}$ is a quantum modular form?
  - In particular, if $\tilde{f}$ is the Eichler integral of a half-integral weight modular form?
  - What about expansions $\tilde{f}(\tau) = \sum B_n (\zeta - q)^n$ at other roots of unity?

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- Inspiring results by Guerzhoy–Kent–Rolen (2014) for Eichler integrals of certain unary theta series.
- In all cases, is it true that the known congruences are complete?

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

S. Ahlgren, B. Kim
Dissections of a "strange" function
Preprint, 2014

G. E. Andrews, J. A. Sellers
Congruences for the Fishburn numbers
Preprint, 2014

F. G. Garvan
Congruences and relations for r-Fishburn numbers
Preprint, 2014

P. Guerzhoy, Z. A. Kent, L. Rolen
Congruences for Taylor expansions of quantum modular forms
Preprint, 2014

A. Straub
Congruences for Fishburn numbers modulo prime powers
Preprint, 2014

Ingredients of proof

- Crucial ingredients: $$F(q) \leftarrow \sum_{n=0}^{N} (q;q)_n = \sum_{i=0}^{p-1} q^i A_p(N, i, q^p)$$

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  If $i \notin S(p)$ and $i_0 \equiv -1/24$ modulo p, then

  $$A_p(pn-1, i, q) = (1-q)^n \alpha_p(n, i, q), \qquad \text{(Andrews–Sellers)}$$
  $$A_p(pn-1, i_0, q) = \left(\frac{12}{p}\right) p q^{\lfloor p/24 \rfloor} F(q^p, pn-1) + (1-q)^n \beta_p(n, q). \qquad \text{(Garvan)}$$

  where $\alpha_p \in \mathbb{Z}[q]$ and $\beta_p \in \mathbb{Z}[q]$.

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  where $\alpha_p \in \mathbb{Z}[q]$ and $\beta_p \in \mathbb{Z}[q]$.
- Ahlgren–Kim (2014) show that, in fact, $(q;q)_n$ divides $A_p(pn-1,i,q)$.

Partition congruences

- Atkin (1968) proved further congruences for small prime moduli.

EG

$$p(13 \cdot 11^3 m + 237) \equiv 0 \pmod{13}$$
$$p(23 \cdot 5^4 m + 3474) \equiv 0 \pmod{23}$$

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- Ono (2000) and Ahlgren–Ono (2001) show that, if M is coprime to 6, then

$$p(Am + B) \equiv 0 \pmod{M}$$

  for infinitely many non-nested arithmetic progressions $Am + B$.
- It is conjectured that no congruences exist for moduli 2 and 3.