Lucas Congruences

Pure Mathematics Seminar
University of South Alabama

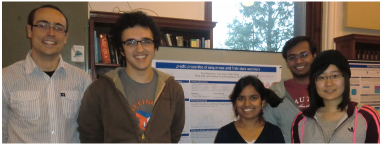
Armin Straub

Oct 16, 2015

University of South Alabama

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$



Arian Daneshvar   Amita Malik   Zhefan Wang
Pujan Dave

(Illinois Geometry Lab, UIUC, Fall 2014)

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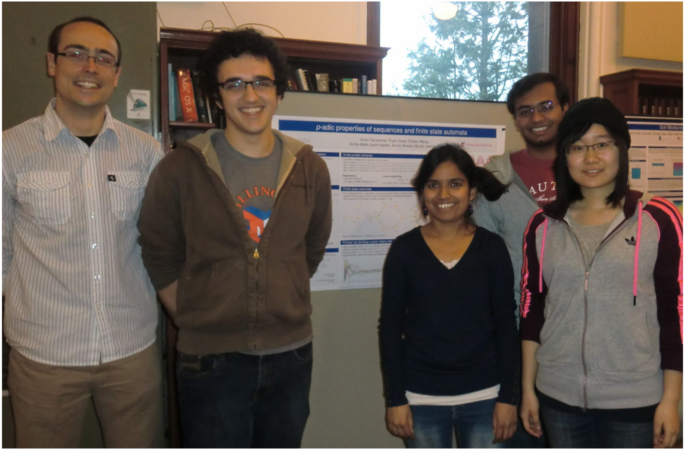

Arian Daneshvar   Amita Malik   Zhefan Wang
Pujan Dave

- semester-long project to introduce undergraduate students to research
- graduate student team leader: Amita Malik

Rough outline

- introducing Apéry-like numbers
- Lucas-type congruences
- applications
  - primes never dividing Apéry-like numbers
  - periodicity modulo p
- a little more on supercongruences (time permitting)

- Let us begin with an open problem:

CONJ Kauers-Zeilberger 2008

All Taylor coefficients of the following function are positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

Positivity of rational functions

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PROP S-Zudilin 2015 The **diagonal coefficients** of the Kauers–Zeilberger function are

$$D(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

- $D(n)$ is an example of an **Apéry-like sequence**.

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Q S-Zudilin 2015

Can we conclude the conjectured positivity from the positivity of $D(n)$ together with the (obvious) positivity of $\frac{1}{1-(x+y+z)+2xyz}$?

The Riemann zeta function

- The **Riemann zeta function** is the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

- Its zeros and values are fundamental, yet mysterious to this day.

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THM Euler 1734:

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots, \qquad \zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$$

CONJ The values $\zeta(3), \zeta(5), \zeta(7), \ldots$ are all transcendental.

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THM Apéry '78 — $\zeta(3)$ is irrational.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

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proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a,b,c) = (17,5,1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}.$$

Q Beukers, Zagier: Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

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Q (Beukers, Zagier): Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right),\qquad \frac{1}{1-C_\alpha z}{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

  with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha=2^4,3^3,2^6,2^4\cdot3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2u_{n+1}=(an^2+an+b)u_n-cn^2u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3u_{n+1}=(2n+1)(an^2+an+b)u_n-n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Apéry-like numbers and modular forms

- The Apéry numbers $A(n)$ satisfy $\qquad 1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad q - 12q^2 + 66q^3 + O(q^4)$ $\qquad q = e^{2\pi i\tau}$

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- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix}\middle| -\frac{1024x}{(1 - x + z)^4}\right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

Lucas congruences

- The Apéry numbers $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy the **Lucas congruences** (Gessel 1982)

$$A(n) \equiv A(n_0) A(n_1) \cdots A(n_r) \pmod{p},$$

where n_i are the p-adic digits of n.

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THM Malik–S 2015: Every sporadic sequence satisfies these Lucas congruences modulo every prime.

Primes not dividing Apéry numbers

CONJ Rowland–Yassawi: There are infinitely many primes p such that p does not divide any Apéry number $A(n)$.
Such as $p = 2, 3, 7, 13, 23, 29, 43, 47, \ldots$

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EG
- The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$.

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- The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$.
- Hence, the Lucas congruences imply that 7 does not divide any Apéry number.

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Primes not dividing Apéry numbers, cont'd

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- Heuristically, combine Lucas congruences,
- palindromic behavior of Apéry numbers, that is

$$A(n) \equiv A(p-1-n) \pmod{p},$$

- and $e^{-1/2} = \lim_{p \to \infty} \left(1 - \frac{1}{p}\right)^{(p+1)/2}$.

Primes not dividing Apéry numbers, cont'd

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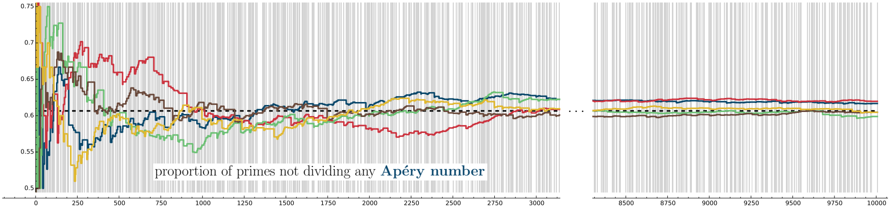

Primes not dividing Apéry numbers, cont'd^2

- The primes below 100 not dividing sporadic sequences, as well as the proportion of primes below $10,000$ not dividing any term

(δ)	2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97	0.6192
(η)	2, 3, 17, 19, 23, 31, 47, 53, 61	0.2897
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THM Malik–S 2015

For any prime $p \neq 3$, we have that, modulo p,

$$A_\eta\left(\left\lfloor \frac{p}{3} \right\rfloor\right) \equiv \begin{cases} (-1)^{\lfloor p/5 \rfloor} \binom{\lfloor p/3 \rfloor}{\lfloor p/15 \rfloor}^3, & \text{if } p \equiv 1, 2, 4, 8 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases}$$

- We therefore expect the proportion of primes not dividing any $A_\eta(n)$ to be $\frac{1}{2} e^{-1/2} \approx 30.33\%$.

THM Ahlgren–Ono '00

The Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

with

$$\sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau).$$

- conjectured by Beukers '87, and proved modulo p
- similar congruences modulo p for other Apéry-like numbers

Periodicity of residues

THM Gessel '82

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even,} \\ 5 \pmod{8}, & \text{if } n \text{ odd.} \end{cases}$$

- conjectured by Chowla–Cowles–Cowles '80
- not eventually periodic modulo 16 (Rowland–Yassawi '13)

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PROP Gessel '82 If $C(n)$ satisfies Lucas congruences modulo p and is eventually periodic modulo p, then

$$C(n) \equiv C(1)^n \pmod p \qquad \text{for all } n = 0, 1, \ldots, p-1.$$

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EG For the Almkvist–Zudilin sequence

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3},$$

$Z(3) - Z(1)^3 = 24$. So can be periodic modulo p only for $p = 2, 3$.

Periodicity of residues

THM DDMSW 2015

The Almkvist–Zudilin numbers

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satisfy the congruences

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- This can be proved using computer algebra in two steps.

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LEM S 2014

The Almkvist–Zudilin numbers are the diagonal coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

That is, $Z(n)$ equals the coefficient of $(x_1x_2x_3x_4)^n$.

Diagonals

- The **diagonal** of a multivariate series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

  is the univariate function $\sum_{n \geqslant 0} a(n, \ldots, n) z^n$.

- The diagonal of an algebraic function is D-finite.

Gessel
Zeilberger
Lipshitz

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$$\frac{1}{1 - x - y}$$

has diagonal coefficients $\binom{2n}{n}$.

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$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}.$$

Finite state automata

EG

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$$\frac{1}{1-x-y} = \sum_{n=0}^{\infty} (x+y)^n$$

has diagonal coefficients $\binom{2n}{n}$. The diagonal is

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}.$$

- The diagonal of a rational function $F(x,y)$ is always algebraic.
  To see this, express the diagonal as $\frac{1}{2\pi i} \int_{|x|=\varepsilon} F(x, \frac{z}{x}) \frac{\mathrm{d}x}{x}$.
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  To see this, express the diagonal as $\frac{1}{2\pi i} \int_{|x|=\varepsilon} F(x, \frac{z}{x}) \frac{\mathrm{d}x}{x}$.
- Not true for more than two variables. However:
  (Furstenberg '67, Deligne '84 and Denef–Lipshitz '87)

THM Diagonals of algebraic functions in $\mathbb{Z}_p[[x_1, \ldots, x_d]]$ are algebraic over $\mathbb{Z}_p(z)$. Equivalently, the diagonal coefficients modulo p^r are generated by a finite state automaton.

Finite state automata

- Recall: the AZ numbers $(-1)^n Z(n)$ are the diagonal coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

Finite state automata

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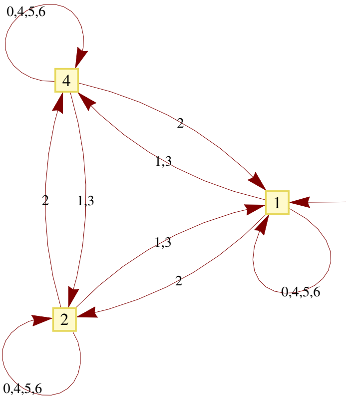
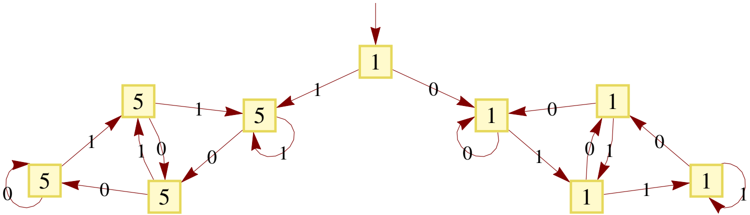
- Rowland–Yassawi (2013) give an algorithm to compute a finite state automaton for these numbers modulo 7 (or any p^r).
- For instance:
$Z(63) = Z(120_{\text{base } 7}) \equiv 1 \pmod{7}$.

Of course, modulo primes it is easier to use Lucas congruences.



- The Almkvist–Zudilin numbers $Z(n)$ modulo 8:

This automatically generated automaton can, of course, be simplified.



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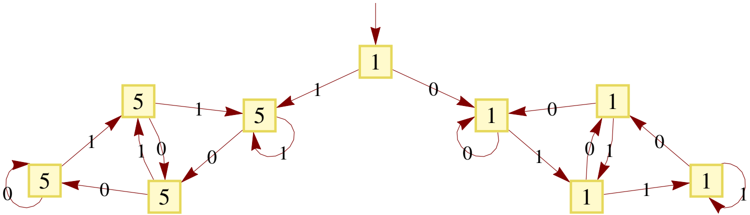

- The automaton makes it obvious that, indeed,

$$Z(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even,} \\ 5 \pmod 8, & \text{if } n \text{ odd.} \end{cases}$$

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,

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THM Beukers, Coster '85, '88 — The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

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EG For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

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- The congruences $a(mp^r) \equiv a(mp^{r-1})$ modulo p^r occur frequently:
  - $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ — Arnold '03, Zarelua '04, ...

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- The congruences $a(mp^r) \equiv a(mp^{r-1})$ modulo p^r occur frequently:
  - $a(n) = \operatorname{tr} A^n$ with $A \in \mathbb{Z}^{d \times d}$ — Arnold '03, Zarelua '04, ...
  - **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$$A(5 \cdot 11^3) = 12488301\ldots\text{about 2000 digits}\ldots\text{about 8000 digits}\ldots 7956522125$$

Weirdly, with this wrong value, one still has

$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \pmod{11^6}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '15
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

Apéry numbers as diagonals

CONJ
Christol
1990

Every holonomic integer sequence with at most exponential growth is the diagonal of a rational function.

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THM S 2014 — The Apéry numbers are the diagonal coefficients of

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$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

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  where $z=\sqrt{1-34x+x^2}$.
- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson

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$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Let $A(\boldsymbol{n})$ be the coefficient of $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} \cdots x_4^{n_4}$.

  Then, for $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

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  Then, for $p \geqslant 5$, we have the **multivariate supercongruences**

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- Numerical evidence suggests the same congruences for

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4}.$$

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- polynomial analogs of Apéry-like numbers
  - find q-analogs (e.g., for Almkvist–Zudilin sequence)
  - q-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - Apéry-like numbers as diagonals and multivariate supercongruences
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Malik, A. Straub
Divisibility properties of sporadic Apéry-like numbers
Preprint, 2015

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

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to appear in Proceedings of the Edinburgh Mathematical Society, 2014

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Positivity of rational functions and their diagonals
Journal of Approximation Theory (special issue dedicated to Richard Askey), Vol. 195, 2015, p. 57-69