Supercongruences for Apéry-like numbers

AKLS seminar on Automorphic Forms
Universität zu Köln

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$1, 5, 73, 1445, 33001, 819005, \ldots$

Includes joint work with:

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

Rough outline

- Introducing Apéry-like numbers
  - they are integer solutions to certain three-term recurrences
  - are all of them known?
- Apéry-like numbers have interesting properties
  - connection to modular forms
  - supercongruences (still open in several cases)
  - multivariate extensions
  - polynomial analogs
- Apéry-like numbers occur in interesting places (if time permits)
  - moments of planar random walks
  - series for $1/\pi$
  - positivity of rational functions
  - counting points on algebraic varieties
  - ...

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

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Beukers, Zagier

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Q **Beukers, Zagier** Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\begin{matrix}\frac{1}{2}, \alpha, 1-\alpha \\ 1, 1\end{matrix}\middle| 4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha, 1-\alpha \\ 1\end{matrix}\middle| \frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

    with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Apéry-like numbers and modular forms

- The Apéry numbers $A(n)$ satisfy $1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i \tau}$

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- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n \geqslant 0} A(n) x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024x}{(1 - x + z)^4} \right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

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THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

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EG For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

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- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \qquad (\operatorname{mod} p^{3r})$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
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$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open!! modulo p^2 Amdeberhan '14
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(C)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\}$$ "points of period n"

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \operatorname{ct} \Lambda(x)^n$ van Straten–Samol '09

  if origin is only interior pt of the Newton polyhedron of $\Lambda(x) \in \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$

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- If $a(1) = 1$, then (C) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions:

s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}\left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n}\right]$$

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CONJ Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3}$$

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Osburn-Sahu-S 2014

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

EG

$$\frac{1}{1 - x - y}$$

has diagonal coefficients $\binom{2n}{n}$.

Diagonals

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- The diagonal of a rational function is D-finite.

Gessel
Zeilberger
Lipshitz

Apéry numbers as diagonals

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The Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}$$

THM
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THM
MacMahon 1915

For $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{m}=(m_1,\ldots,m_n)\in\mathbb{Z}_{\geqslant 0}^n$,

$$[\boldsymbol{x}^{\boldsymbol{m}}]\frac{1}{\det(I_n-BX)}=[\boldsymbol{x}^{\boldsymbol{m}}]\prod_{i=1}^n\left(\sum_{j=1}^n B_{i,j}x_j\right)^{m_i},$$

where $B\in\mathbb{C}^{n\times n}$ and X is the diagonal matrix with entries $x_1,\ldots,x_n$.

Apéry numbers as diagonals

THM S 2014

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$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}^4_{\geqslant 0}} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

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where $B \in \mathbb{C}^{n\times n}$ and X is the diagonal matrix with entries $x_1, \ldots, x_n$.

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4 \end{pmatrix}$$

$$A(\boldsymbol{n}) = [\boldsymbol{x}^{\boldsymbol{n}}](x_1+x_2+x_3)^{n_1}(x_1+x_2)^{n_2}(x_3+x_4)^{n_3}(x_2+x_3+x_4)^{n_4}$$

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- The coefficients are the multivariate Apéry numbers

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- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- Such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even,} \\ 5 \pmod 8, & \text{if } n \text{ odd.} \end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

Multivariate supercongruences

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Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

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- $\sum_{n\geqslant 0} a(n)x^n = F(x) \quad \Longrightarrow \quad \sum_{n\geqslant 0} a(pn)x^{pn} = \frac{1}{p}\sum_{k=0}^{p-1} F(\zeta_p^k x)$ $\qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n}p^r)$ and $A(\boldsymbol{n}p^{r-1})$ have rational generating function.
  The proof, however, relies on an explicit binomial sum for the coefficients.

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- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}.$$

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- Because $A(n-1) = A(-n, -n, -n, -n)$, we also find

$$A(m p^r - 1) \equiv A(m p^{r-1} - 1) \pmod{p^{3r}}.$$

Beukers '85

Many more conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

  $1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$

$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$

$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$

$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$

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$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

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CONJ S 2014 The coefficients $B(\boldsymbol{n})$ of each of these satisfy, for $p \geqslant 5$,

$$B(\boldsymbol{n}p^r) \equiv B(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

An infinite family of rational functions

THM
S 2014

Let $\lambda \in \mathbb{Z}_{>0}^{\ell}$ with $d = \lambda_1 + \ldots + \lambda_\ell$. Define $A_\lambda(\boldsymbol{n})$ by

$$\frac{1}{\prod_{1 \leqslant j \leqslant \ell} \left[1 - \sum_{1 \leqslant r \leqslant \lambda_j} x_{\lambda_1 + \ldots + \lambda_{j-1} + r}\right] - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

EG

$\lambda = (2, 2)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}$$

$\lambda = (2, 1)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3) - x_1 x_2 x_3}$$

Further examples

EG

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}$$

has as diagonal the Apéry-like numbers, associated with $\zeta(2)$,

$$B(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

EG

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$$

has as diagonal the numbers

$d=3$: Franel, $d=4$: Yang–Zudilin

$$Y_d(n)=\sum_{k=0}^{n}\binom{n}{k}^d.$$

- In each case, we obtain supercongruences generalizing results of Coster (1988) and Chan–Cooper–Sica (2010).

A conjectural multivariate supercongruence

CONJ S 2014 The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open.

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

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- The q-factorial:

$$[n]_q! = [n]_q \, [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} = \binom{n}{n-k}_q$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

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$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

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- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q = 1$
  and hence invisible in the classical world

- Here's some q-binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal. Sylvester, 1878

A few faces of the q-binomial coefficient

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{j}_q = \binom{n-1}{j-1}_q + q^j \binom{n-1}{j}_q$, The q-binomial coefficient $\binom{n}{k}_q$

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- has a q-integral representation analogous to the beta function,
- counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2p}{p}_q = \sum_k \binom{p}{k}_q \binom{p}{p-k}_q q^{(p-k)^2}$$

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THM Clark 1995

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- Similar results by Andrews; e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM
S 2011

For any prime p,

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - (a-b)b\binom{a}{b}\frac{p^2-1}{24}(q^p-1)^2 \pmod{[p]_q^3}.$$

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EG Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132}$ is an irreducible polynomial with integer coefficients.

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  Is there a nice explanation or analog in the q-world?

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  Is there a nice explanation or analog in the q-world?
- The congruence holds mod $\Phi_n(q)^3$ if p is replaced by any integer n.
  (No classical counterpart since $\Phi_n(1)=1$ unless n is a prime power.)

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin '06

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- Appear implicitly in work of Krattenthaler–Rivoal–Zudilin '06
- The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

THM
S 2015

The q-Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfy the supercongruences

$$A_q(pn) \equiv A_{q^{p^2}}(n) - \frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

THM
S 2015

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satisfy the supercongruences

$$A_q(pn) \equiv A_{q^{p^2}}(n) - \frac{p^2-1}{12}(q^p-1)^2 f(n) \pmod{[p]_q^3}.$$

- The numbers $f(n)$ can be expressed as $0, 5, 292, 13005, 528016, \ldots$

$$f(n) = \sum_{k=0}^{n} g(n,k) \binom{n}{k}^2 \binom{n+k}{k}^2, \qquad g(n,k) = k(2n-k) + \frac{k^4}{(n+k)^2}.$$

- Similar q-analogs and congruences for other Apéry-like numbers?

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- polynomial analogs of Apéry-like numbers
  - find q-analogs (e.g., for Almkvist–Zudilin sequence)
  - q-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

A. Straub
Multivariate Apéry numbers and supercongruences of rational functions
Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

R. Osburn, B. Sahu, A. Straub
Supercongruences for sporadic sequences
to appear in Proceedings of the Edinburgh Mathematical Society, 2014

A. Straub, W. Zudilin
Positivity of rational functions and their diagonals
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2014

M. Rogers, A. Straub
A solution of Sun's $520 challenge concerning $520/\pi$
International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288

J. Borwein, A. Straub, J. Wan, W. Zudilin (appendix by D. Zagier)
Densities of short uniform random walks
Canadian Journal of Mathematics, Vol. 64, Nr. 5, 2012, p. 961-990

A. Straub
A q*-analog of Ljunggren's binomial congruence*
DMTCS Proceedings: FPSAC 2011, p. 897-902

Applications of Apéry-like numbers

- Random walks

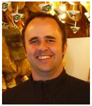
Jon Borwein
U. of Newcastle, AU

Dirk Nuyens
K.U.Leuven, BE

James Wan
SUTD, SG

Wadim Zudilin
U. of Newcastle, AU

- Series for $1/\pi$



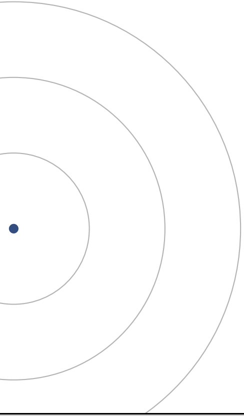
Mat Rogers
U. of Montreal, CA

- Positivity of rational functions

Wadim Zudilin
U. of Newcastle, AU

Example I: Random walks

n steps in the plane
(length 1, random direction)



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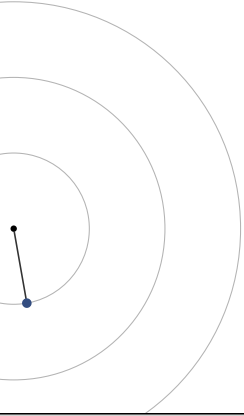

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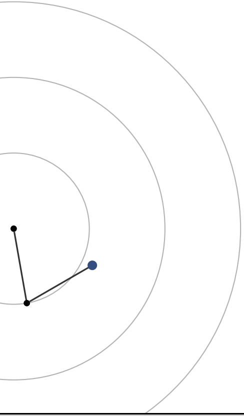

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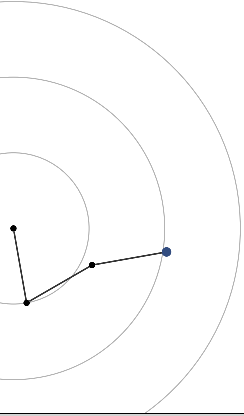

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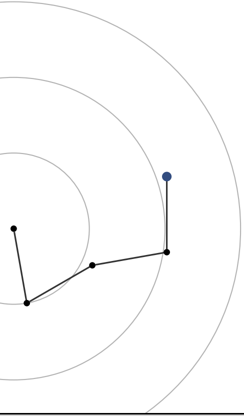

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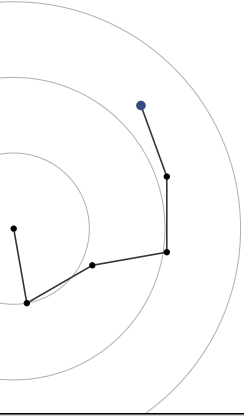

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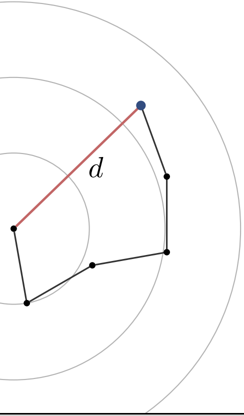

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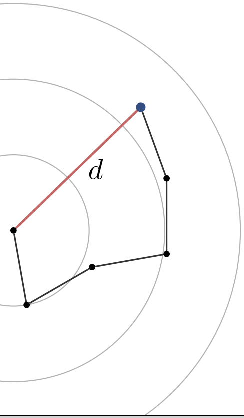
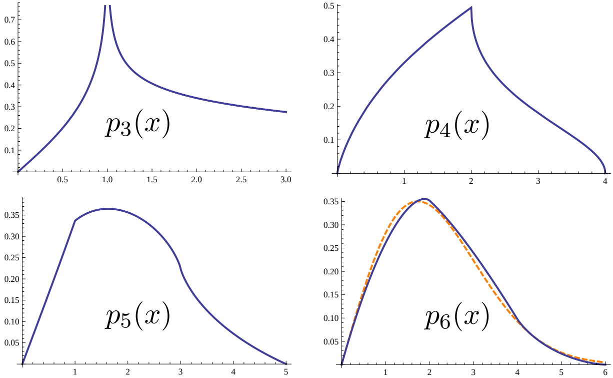


- $p_n(x)$ — probability density of distance traveled



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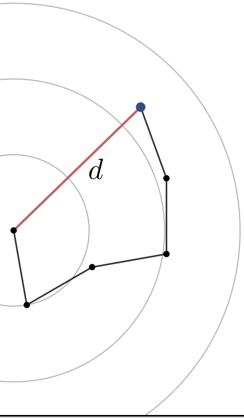

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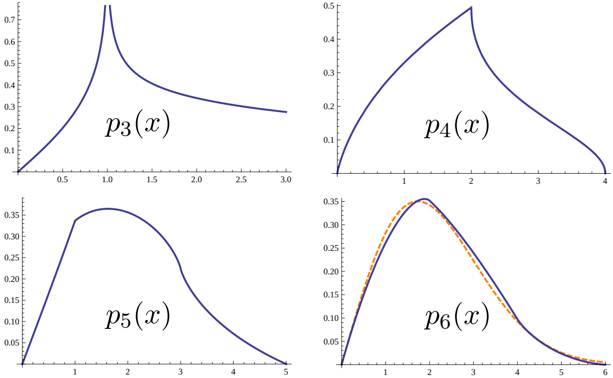

- $W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$ — probability moments

$$W_2(1) = \frac{4}{\pi}, \qquad W_3(1) = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right)$$

classical — Borwein–Nuyens–S–Wan, 2010

Example I: Random walks

- The probability moments

$$W_n(s) = \int_0^\infty x^s p_n(x)\,\mathrm{d}x$$

include the Apéry-like numbers

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j},$$

$$W_4(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \binom{2(k-j)}{k-j}.$$

Example I: Random walks

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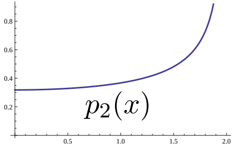
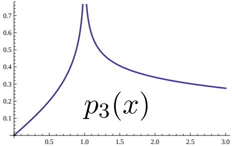
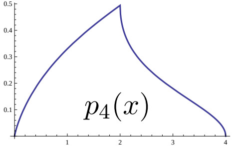
THM Borwein-Nuyens-S-Wan 2010

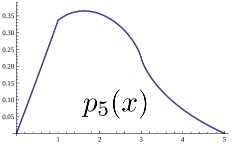
$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2$$

Example I: Random walks









$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

easy

$$p_3(x) = \frac{2\sqrt{3}}{\pi} \frac{x}{(3+x^2)} {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}\middle| \frac{x^2\left(9-x^2\right)^2}{\left(3+x^2\right)^3}\right)$$

classical
with a spin

$$p_4(x) = \frac{2}{\pi^2} \frac{\sqrt{16-x^2}}{x} \operatorname{Re} {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{6}, \frac{7}{6}\end{matrix}\middle| \frac{\left(16-x^2\right)^3}{108x^4}\right)$$

new
BSWZ 2011

$$p_5'(0) = \frac{\sqrt{5}}{40\pi^4}\Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15}) \approx 0.32993$$

Example II: Series for $1/\pi$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (6n+1) \frac{1}{4^n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}}$$

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

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$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{1103+26390n}{396^{4n}}$$

- Last series used by Gosper in 1985 to compute $17,526,100$ digits of π
- First proof of all of Ramanujan's 17 series by Borwein brothers

Srinivasa Ramanujan
Modular equations and approximations to π
Quart. J. Math., Vol. 45, p. 350–372, 1914

Jonathan M. Borwein and Peter B. Borwein
Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity
Wiley, 1987

- Sato observed that series for $\frac{1}{\pi}$ can be built from Apéry-like numbers:

EG Chan-Chan-Liu 2003

For the Domb numbers $D(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$,

$$\frac{8}{\sqrt{3}\pi}=\sum_{n=0}^{\infty}D(n)\frac{5n+1}{2^{6n}}.$$

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- Sun offered a \$520 bounty for a proof the following series:

THM Rogers-S 2012

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

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FACT

- For $\tau \in \mathbb{Q}(\sqrt{-d})$, $x(\tau)$ is an algebraic number.
- $f'(\tau)$ is a **quasimodular** form.
- Prototypical $E_2(\tau)$ satisfies $\tau^{-2} E_2\left(-\frac{1}{\tau}\right) - E_2(\tau) = \dfrac{6}{\pi i \tau}$.

- These are the main ingredients for series for $1/\pi$. Mix and stir.

Example III: Positivity of rational functions

- A rational function

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a_{n_1, \ldots, n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is **positive** if $a_{n_1, \ldots, n_d} > 0$ for all indices.

EG The following rational functions are positive.

$$S(x, y, z) = \frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + yz + zx)}$$

Szegő '33
Kaluza '33
Askey–Gasper '72
S '08

$$A(x, y, z) = \frac{1}{1 - (x + y + z) + 4xyz}$$

Askey–Gasper '77
Koornwinder '78
Ismail–Tamhankar '79
Gillis–Reznick–Zeilberger '83

- Both functions are on the boundary of positivity.

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- Both functions are on the boundary of positivity.
- The diagonal coefficients of A are the **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$.

Example III: Positivity of rational functions

CONJ Kauers-Zeilberger 2008 The following rational function is positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

- Would imply conjectured positivity of Lewy–Askey rational function

$$\frac{1}{1-(x+y+z+w)+\frac{2}{3}(xy+xz+xw+yz+yw+zw)}.$$

Recent proof of non-negativity by Scott and Sokal, 2013

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Recent proof of non-negativity by Scott and Sokal, 2013

PROP S-Zudilin 2013

The Kauers–Zeilberger function has diagonal coefficients

$$d_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2.$$

Positivity of rational functions

- Consider rational functions $F = 1/p(x_1, \ldots, x_d)$ with p a symmetric polynomial, linear in each variable.

Q Under what condition(s) is the positivity of F implied by the positivity of its diagonal?

EG
- $\frac{1}{1-(x+y)}$ is positive.
- $\frac{1}{1+x+y}$ has positive diagonal but is not positive.

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Q F positive $\Longleftrightarrow$ diagonal of F and $F|_{x_d=0}$ are positive?

THM S-Zudilin 2013

$$F(x,y) = \frac{1}{1 + c_1(x+y) + c_2 xy} \quad \text{is positive}$$

$\Longleftrightarrow$ diagonal of F and $F|_{y=0}$ are positive

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

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Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008

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to appear in Proceedings of the Edinburgh Mathematical Society, 2014

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