Core partitions into distinct parts and an analog of Euler's theorem

Integers Conference 2016
University of West Georgia

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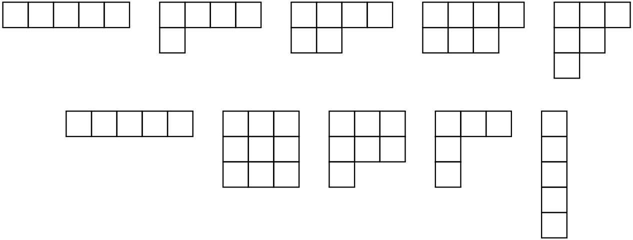
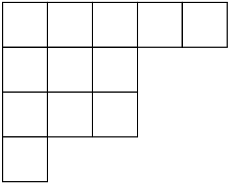
Oct 6, 2016

University of South Alabama



Core partitions

- The integer partition $(5, 3, 3, 1)$ has Young diagram:



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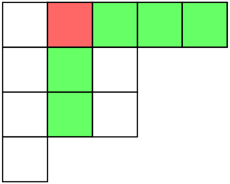

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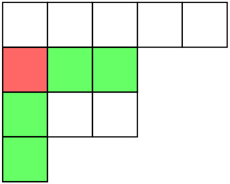

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LEM If a partition is t-core, then it is also rt-core for $r = 1, 2, 3 \ldots$

The number of core partitions

- Using the theory of modular forms, Granville and Ono (1996) showed:

(The case $t = p$ of this completed the classification of simple groups with defect zero Brauer p-blocks.)

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- If $c_t(n)$ is the number of t-core partitions of n, then

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}.$$

$$\sum_{n=0}^{\infty} c_2(n) q^n = \sum_{n=0}^{\infty} q^{\frac{1}{2}n(n+1)}, \quad \sum_{n=0}^{\infty} c_3(n) q^n = 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + \ldots$$

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COR The total number of t-core partitions is infinite.

Though this is probably the most complicated way possible to see that...

Counting core partitions

THM Anderson 2002

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- Note that the number of $(s,s+1)$-core partitions is the Catalan number

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- Ford, Mai and Sze (2009) show that the number of self-conjugate (s,t)-core partitions is

$$\binom{\lfloor s/2\rfloor + \lfloor t/2\rfloor}{\lfloor s/2\rfloor}.$$

Core partitions into distinct parts

- Amdeberhan raises the interesting problem of counting the number of special partitions which are t-core for certain values of t.

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- He further conjectured that the largest possible size of an $(s, s+1)$-core partition into distinct parts is $\lfloor s(s+1)/6 \rfloor$, and that there is a unique such largest partition unless $s \equiv 1$ modulo 3, in which case there are two partitions of maximum size.
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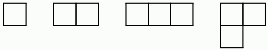

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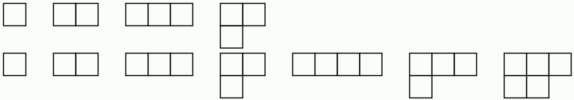
EG

$s=4$, $F_5=5$: $\emptyset$

$s=5$, $F_6=8$: $\emptyset$



A two-parameter generalization

THM
S 2016

Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1) = 1$, $N_d(2) = d$ and

$$N_d(s) = N_d(s-1) + dN_d(s-2).$$

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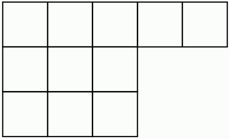
EG The first few generalized Fibonacci polynomials $N_d(s)$ are

$$1, \quad d, \quad 2d, \quad d(d+2), \quad d(3d+2), \quad d(d^2+5d+2), \ldots$$

For $d = 1$, we recover the usual Fibonacci numbers.
For $d = 2$, we find $N_2(s) = 2^{s-1}$.

DEF The **perimeter** of a partition is the maximum hook length in λ.

EG The partition  has perimeter 7.

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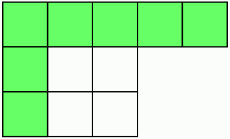
EG The partition  has perimeter 7.

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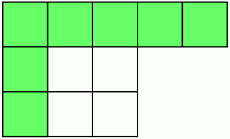
EG The partition  has perimeter 7.

- Introduced (up to a shift by 1) by Corteel and Lovejoy (2004) in their study of overpartitions.
- The perimeter is the largest part plus the number of parts (minus 1).
- The **rank** is the largest part minus the number of parts.

An analog of Euler's theorem

THM
S 2016

The number of partitions into distinct parts with perimeter M equals the number of partitions into odd parts with perimeter M.

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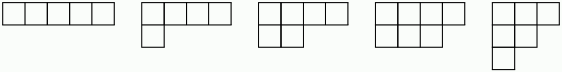

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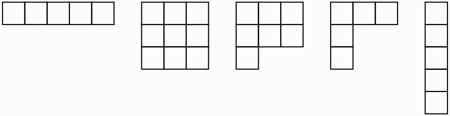

- While it appears natural and is easily proved, we have been unable to find this result in the literature.

An analog of Euler's theorem

THM S 2016 — The number of partitions into distinct parts with perimeter M equals the number of partitions into odd parts with perimeter M. Both are enumerated by the Fibonacci number F_M.

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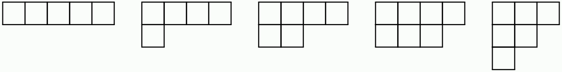

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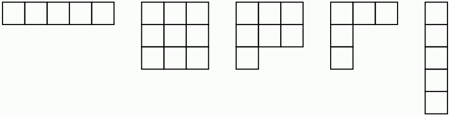

In each case, there are $F_5 = 5$ many of these partitions.

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Euler's theorem

THM The number $D(n)$ of partitions of n into distinct parts equals the number $O(n)$ of partitions of n into odd parts.

proof Euler famously proved his claim using a very elegant manipulation of generating functions:

$$\begin{aligned}\sum_{n\geqslant 0} D(n)x^n &= (1+x)(1+x^2)(1+x^3)\cdots \\ &= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\cdots \\ &= \frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots = \sum_{n\geqslant 0} O(n)x^n.\end{aligned}$$

□

- Bijective proofs for instance by Sylvester.

Refinements of Euler's theorem

- Bousquet-Mélou and Eriksson (1997): the number of partitions of n into distinct parts with sign-alternating sum k is equal to the number of partitions of n into k odd parts.
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Q Do similarly interesting refinements exist for partitions into distinct (respectively odd) parts with perimeter M?

- Fu and Tang (2016) indeed prove refinements analogous to Fine's.

  The number of partitions with perimeter n into distinct parts with maximum part M is equal to the number of partitions with perimeter n into odd parts such that the maximum part plus twice the number of parts is $2M+1$.

Partitions of bounded perimeter

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COR An $(s, ds-1)$-core partition into distinct parts has perimeter at most $ds-2$.

Summary

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The number of (s,t)-core partitions is finite if and only if s and t are coprime. In that case, this number is

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Let $N_d(s)$ be the number of $(s, ds-1)$-core partitions into distinct parts. Then, $N_d(1) = 1$, $N_d(2) = d$ and

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- In particular, there are F_s many $(s-1,s)$-core partitions into distinct parts,
- and 2^{s-1} many $(s, 2s-1)$-core partitions into distinct parts.

Q What is the number of (s,t)-core partitions into distinct parts in general?

Enumerating (s,t)-core partitions into distinct parts

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$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
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12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

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10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
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12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

Q What is the number of (s,t)-core partitions into distinct parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
11	1	6	8	15	21	32	64	101	256	89	∞	144
12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

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1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
9	1	5	∞	11	16	∞	64	34	∞	55	256	∞
10	1	∞	7	∞	∞	∞	50	∞	55	∞	89	∞
11	1	6	8	15	21	32	64	101	256	89	∞	144
12	1	∞	∞	∞	38	∞	114	∞	∞	∞	144	∞

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1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
									∞	55	256	∞
									55	∞	89	∞
									56	89	∞	144
									∞	∞	144	∞

CONJ If s is odd, there are 2^{s-1} many $(s,s+2)$-core partitions into distinct parts.

Yan, Qin, Jin, Zhou (2016) have very recently proven this conjecture by analyzing order ideals in an associated poset introduced by Anderson.

Enumerating (s,t)-core partitions into distinct parts

Q What is the number of (s,t)-core partitions into distinct parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	∞	2	∞	3	∞	4	∞	5	∞	6	∞
3	1	2	∞	3	4	∞	5	6	∞	7	8	∞
4	1	∞	3	∞	5	∞	8	∞	11	∞	15	∞
5	1	3	4	5	∞	8	16	18	16	∞	21	38
6	1	∞	∞	∞	8	∞	13	∞	∞	∞	32	∞
7	1	4	5	8	16	13	∞	21	64	50	64	114
8	1	∞	6	∞	18	∞	21	∞	34	∞	101	∞
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CONJ If s is odd, there are 2^{s-1} many $(s,s+2)$-core partitions into distinct parts.

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$(s, s+3)$-core partitions into distinct parts

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

- The largest size of $(2n-1, 2n+1)$-core partitions into distinct parts is
$$\frac{1}{24}n(n^2-1)(5n+6).$$
Now, also proven by Yan, Qin, Jin, Zhou (2016).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$

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- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$
- The largest size of $(3n-2, 3n+1)$-core partitions into distinct parts appears to be
$$\frac{1}{24} n(n^2-1)(9n+10).$$

$(s, s+3)$-core partitions into distinct parts

THM 2^{s-1} many $(s, s+2)$-core partitions into distinct parts (s odd).

- The largest size of $(2n-1, 2n+1)$-core partitions into distinct parts is
$$\frac{1}{24}n(n^2-1)(5n+6).$$
Now, also proven by Yan, Qin, Jin, Zhou (2016).

Q How many $(s, s+3)$-core partitions into distinct parts?

- $1, 3, \infty, 8, 18, \infty, 50, 101, \infty, 291, 557, \infty, 1642, 3048, \infty, 9116, 16607, \ldots$
- The largest size of $(3n-2, 3n+1)$-core partitions into distinct parts appears to be
$$\frac{1}{24}n(n^2-1)(9n+10).$$
- The largest size of $(3n-1, 3n+2)$-core partitions into distinct parts appears to be
$$\frac{1}{24}n(9n^3+38n^2+39n-14).$$

Enumerating (s,t)-core partitions into odd parts

Q What is the number of (s,t)-core partitions into odd parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2
3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
10	1	2	8	10	∞	34	78	78	404	∞	790	308
11	1	2	8	13	41	62	87	124	166	790	∞	2140
12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

Enumerating (s,t)-core partitions into odd parts

Q What is the number of (s,t)-core partitions into odd parts?

$s\backslash t$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2	2
3	1	2	∞	4	4	∞	6	6	∞	8	8	∞
4	1	2	4	∞	7	6	9	∞	11	10	13	∞
5	1	2	4	7	∞	17	12	17	25	∞	41	31
6	1	2	∞	6	17	∞	31	21	∞	34	62	∞
7	1	2	6	9	12	31	∞	80	43	78	87	97
8	1	2	6	∞	17	21	80	∞	152	78	124	∞
9	1	2	∞	11	25	∞	43	152	∞	404	166	∞
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12	1	2	∞	∞	31	∞	97	∞	∞	308	2140	∞

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

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