Divisibility properties of sporadic Apéry-like numbers

Special Session on Experimental Mathematics
AMS Spring Southeastern Sectional Meeting, Athens

Armin Straub

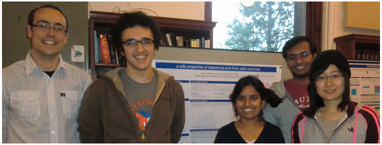
Mar 6, 2016

University of South Alabama

based on joint work with Amita Malik

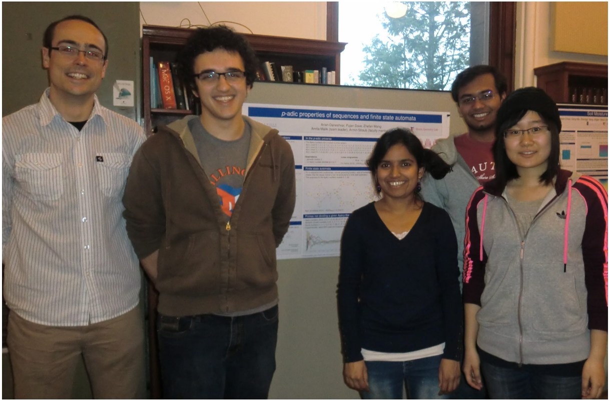
$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$



Arian Daneshvar   Amita Malik   Zhefan Wang
Pujan Dave

(Illinois Geometry Lab, UIUC, Fall 2014)



Arian Daneshvar **Amita Malik** Zhefan Wang
Pujan Dave

- semester-long project to introduce undergraduate students to research
- graduate student team leader: Amita Malik

Rough outline

- introducing Apéry-like numbers
- Lucas-type congruences
- applications

Positivity of rational functions

- Let us begin with an open problem:

CONJ Kauers-Zeilberger 2008

All Taylor coefficients of the following function are positive:

$$\frac{1}{1-(x+y+z+w)+2(yzw+xzw+xyw+xyz)+4xyzw}.$$

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PROP S-Zudilin 2015

The **diagonal coefficients** of the Kauers–Zeilberger function are

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- $D(n)$ is an example of an **Apéry-like sequence**.

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Q S-Zudilin 2015 Can we conclude the conjectured positivity from the positivity of $D(n)$ together with the (obvious) positivity of $\frac{1}{1-(x+y+z)+2xyz}$?

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q (Beukers, Zagier): Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Essentially, only 14 tuples (a, b, c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \alpha, 1-\alpha \\ 1, 1\end{matrix}\right| 4C_\alpha z\right), \qquad \frac{1}{1 - C_\alpha z}\,{}_2F_1\left(\left.\begin{matrix}\alpha, 1-\alpha \\ 1\end{matrix}\right| \frac{-C_\alpha z}{1 - C_\alpha z}\right)^2,$$

  with $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4 \cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - n(cn^2 + d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Lucas congruences

- The Apéry numbers $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy the **Lucas congruences** (Gessel 1982)

$$A(n) \equiv A(n_0) A(n_1) \cdots A(n_r) \pmod{p},$$

where n_i are the p-adic digits of n.

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THM Malik–S 2015: Every (known) sporadic sequence satisfies these Lucas congruences modulo every prime.

Approaches to proving Lucas congruences

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EG For the Apéry numbers, $\Lambda(x, y, z) = \frac{(x+y)(z+1)(x+y+z)(y+z+1)}{xyz}$.

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- Neither of these works for the sequence $A_\eta(n) = \operatorname{ct} \Lambda(x, y, z)^n$, where

$$\Lambda(x, y, z) = \left(1 - \frac{1}{xy(1+z)^5}\right) \frac{(1+x)(1+y)(1+z)^4}{z^3}.$$

A crucial ingredient of our proof is a technique used by Calkin (1998) to prove that $\sum_k \binom{n}{k}^{2a}$ is divisible by all primes p with $n < p < n + 1 + \frac{n}{2a-1}$.

Primes not dividing Apéry numbers

CONJ Rowland–Yassawi: There are infinitely many primes p such that p does not divide any Apéry number $A(n)$.
Such as $p = 2, 3, 7, 13, 23, 29, 43, 47, \ldots$

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- The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$.

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- The values of Apéry numbers $A(0), A(1), \ldots, A(6)$ modulo 7 are $1, 5, 3, 3, 3, 5, 1$.
- Hence, the Lucas congruences imply that 7 does not divide any Apéry number.

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- Heuristically, combine Lucas congruences,
- palindromic behavior of Apéry numbers, that is

$$A(n) \equiv A(p-1-n) \pmod{p},$$

- and $e^{-1/2} = \lim_{p\to\infty} \left(1 - \frac{1}{p}\right)^{(p+1)/2}$.

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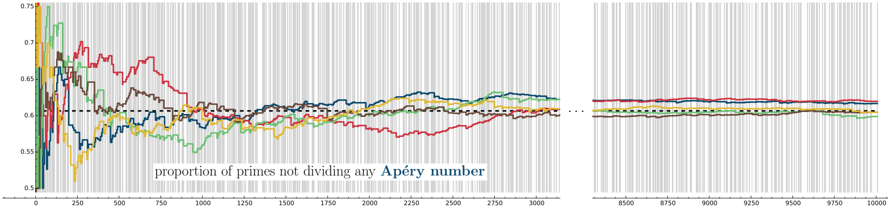

Primes not dividing Apéry numbers, cont'd[2]

- The primes below 100 not dividing sporadic sequences, as well as the proportion of primes below $10,000$ not dividing any term

(δ)	2, 5, 7, 11, 13, 19, 29, 41, 47, 61, 67, 71, 73, 89, 97	0.6192
(η)	2, 3, 17, 19, 23, 31, 47, 53, 61	0.2897
(α)	3, 5, 13, 17, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 83, 89	0.5989
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THM Malik–S 2015

For any prime $p \neq 3$, we have that, modulo p,

$$A_\eta\left(\left\lfloor \frac{p}{3} \right\rfloor\right) \equiv \begin{cases} (-1)^{\lfloor p/5 \rfloor} \binom{\lfloor p/3 \rfloor}{\lfloor p/15 \rfloor}^3, & \text{if } p \equiv 1, 2, 4, 8 \pmod{15}, \\ 0, & \text{otherwise.} \end{cases}$$

- We therefore expect the proportion of primes not dividing any $A_\eta(n)$ to be $\frac{1}{2} e^{-1/2} \approx 30.33\%$.

Modular (super)congruences

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THM Stienstra–Beukers 1985

For any prime $p \neq 2$, we have that, modulo p,

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THM Ahlgren 2001

$$A_b\left(\left\lfloor \frac{p}{2} \right\rfloor\right) \equiv c_p \pmod{p^2},$$

where c_p are the Fourier coefficients of the modular form

$$\eta(4z)^6 := q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = \sum_{n=1}^{\infty} c_n q^n, \quad q = e^{2\pi i z}.$$

Similar congruences by Ahlgren and Ono (2000) for the Apéry numbers.

Apéry-like numbers and modular forms

- The Apéry numbers $A(n)$ satisfy $\qquad 1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad q - 12q^2 + 66q^3 + O(q^4)$ $\qquad q = e^{2\pi i \tau}$

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- As a consequence, with $z=\sqrt{1-34x+x^2}$,

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right).$$

- Context:

$f(\tau)$	modular form of (integral) weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau))=f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

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THM Beukers, Coster '85, '88

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EG

For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

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  - **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$$A(5 \cdot 11^3) = 12488301 \ldots \text{about 2000 digits} \ldots \text{about 8000 digits} \ldots 795652125$$

Weirdly, with this wrong value, one still has

$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \pmod{11^6}.$$

Supercongruences for Apéry-like numbers

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- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '14
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '15
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \left(\binom{4n-5k-1}{3n} + \binom{4n-5k}{3n} \right)$	Osburn–Sahu–S '14
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	open

Some of many open problems

- Supercongruences for all Apéry-like numbers
  - proof of all the classical ones
  - uniform explanation, proofs not relying on binomial sums
- Apéry-like numbers as diagonals
  - find minimal rational functions
  - extend supercongruences
  - any structure?
- polynomial analogs of Apéry-like numbers
  - find q-analogs (e.g., for Almkvist–Zudilin sequence)
  - q-supercongruences
  - is there a geometric picture?
- Many further questions remain.
  - is the known list complete?
  - Apéry-like numbers as diagonals and multivariate supercongruences
  - higher-order analogs, Calabi–Yau DEs
  - modular supercongruences — Beukers '87, Ahlgren–Ono '00

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}, \qquad \sum_{n=1}^{\infty} a(n) q^n = \eta^4(2\tau)\eta^4(4\tau)$$

  - ...

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

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