Congruences connecting modular forms and truncated hypergeometric series

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$${}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}$$

Joint work with:

Robert Osburn
(University College Dublin)

Wadim Zudilin
(University of Newcastle/
Radboud Universiteit)

The wonderful world of $A = B$

EG

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} = \sum_{k=0}^{n} (-1)^{n+k} \binom{3n+1}{n-k} \binom{n+k}{k}^3$$

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EG
Apéry '78

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the difference equation

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5) u_n - n^3 u_{n-1}.$$

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$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

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EG

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(1 - 2k(2H_k - H_{n+k} - H_{n-k})\right) = 1$$

Scott Ahlgren, Shalosh B. Ekhad, Ken Ono, Doron Zeilberger
A binomial coefficient identity associated to a conjecture of Beukers
Electronic Journal of Combinatorics, Vol. 5, 1998, #R10

EG again

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- Below, $p > 2$ is a prime and $n = (p-1)/2$.

EG OSZ 2017

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 \binom{n+k}{k}^3 \left(1 - 3k(2H_k - H_{n+k} - H_{n-k})\right) \equiv \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \pmod{p^2}$$

EG again

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (1 - 2k(2H_k - H_{n+k} - H_{n-k})) = 1$$

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EG OSZ 2017

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EG OSZ 2017

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \equiv (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \pmod{p^2}$$

- We have no general algorithmic approach to such congruences.
- Instead, we had to find suitable intermediate **identities**.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

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THM (Apéry '78) $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Hypergeometric series

EG Trivially, the Apéry numbers have the representation

$$\begin{aligned} A(n) &= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \\ &= {}_4F_3\left(\begin{matrix} -n, -n, n+1, n+1 \\ 1, 1, 1 \end{matrix} \middle| 1\right). \end{aligned}$$

- Here, ${}_4F_3$ is a hypergeometric series:

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!}.$$

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- Similary, we have the **truncated hypergeometric series**

$$ {}_pF_q\left(\begin{matrix}a_1, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}\middle| z\right)_M = \sum_{k=0}^{M} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^n}{n!}.$$

A first connection to modular forms

- The Apéry numbers $A(n)$ satisfy $1, 5, 73, 1145, \ldots$

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}} .$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad q - 12q^2 + 66q^3 + O(q^4)$ $\quad q = e^{2\pi i\tau}$

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\qquad$ $q = e^{2\pi i\tau}$

EG As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n \geqslant 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}} \, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix} \middle| -\frac{1024x}{(1 - x + z)^4}\right).$$

EG
S 2014 For contrast, the Apéry numbers are the diagonal coefficients of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1x_2x_3x_4}.$$

A second connection to modular forms

THM Ahlgren–Ono '00

For primes $p > 2$, the Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2}$$

where $a(n)$ are the Fourier coefficients of the Hecke eigenform

$$\eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n) q^n$$

of weight 4 for the modular group $\Gamma_0(8)$.

- conjectured by Beukers '87, and proved modulo p
- similar congruences modulo p for other Apéry-like numbers

The “super” in these congruences

Fourier coefficients $a(p)$

Apéry sequence $A(n)$

The “super” in these congruences

Fourier coefficients $a(p)$

$\downarrow\uparrow$

point counts on modular curves modulo p

$\downarrow\uparrow$

character sums

$\downarrow\uparrow$

Gaussian hypergeometric series

$\downarrow\uparrow$

harmonic sums

$\downarrow\uparrow$

truncated hypergeometric series

$\downarrow\uparrow$

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equalities

"easy" mod p

Kilbourn's extension of the Ahlgren–Ono supercongruence

THM
Kilbourn 2006

$$ {}_4F_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}\middle| 1\right)_{p-1} \equiv a(p) \pmod{p^3}, $$

for primes $p > 2$. Again, $a(n)$ are the Fourier coefficients of

$$ \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n)q^n. $$

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$$ {}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv a(p) \pmod{p^3}, $$

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$$ \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n=1}^{\infty} a(n) q^n. $$

- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series ${}_4F_3$ and
  - Fourier coefficients of modular forms of weight 4.
- Despite considerable progress, 11 of these remain open.

  McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves "half" of each of the 14.
  2017/5/4: Preprint by Long–Tu–Yui–Zudilin proving all 14 congruences.
- The 14 supercongruence conjectures were complemented with $4+4$ conjectures for ${}_2F_1$ and ${}_3F_2$.

A supercongruence for ${}_6F_5$

THM
OSZ 2017

$$ {}_6F_5\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1, 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $b(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n=1}^{\infty} b(n)q^n, $$

the unique newform in $S_6(\Gamma_0(8))$.

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.

A supercongruence for ${}_6F_5$

THM
OSZ 2017

$$ {}_6F_5\left(\begin{matrix}\frac12, \frac12, \frac12, \frac12, \frac12, \frac12\\ 1, 1, 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}, $$

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the unique newform in $S_6(\Gamma_0(8))$.

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.
- A result of Frechette, Ono and Papanikolas expresses the $b(p)$ in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo p^3 in terms of sums involving harmonic sums.

A brief impression of the available ingredients

THM In terms of Gaussian hypergeometric series,

$$b(p) = -p^5 {}_6F_5(1) + p^4 {}_4F_3(1) + p^3 {}_2F_1(1) + p^2.$$

- Conjectured by Koike; proven by Frechette, Ono and Papanikolas (2004).
- Here, ϕ_p is the quadratic character mod p, ϵ_p the trivial character, and

$$ {}_{n+1}F_n(x) = {}_{n+1}F_n\left(\begin{matrix}\phi_p,\, \phi_p,\, \ldots,\, \phi_p \\ \epsilon_p,\, \ldots,\, \epsilon_p\end{matrix} \,\middle|\, x\right)_p,$$

  the finite field version of

$$ {}_{n+1}F_n\left(\begin{matrix}\frac{1}{2},\, \frac{1}{2},\, \ldots,\, \frac{1}{2} \\ 1,\, \ldots,\, 1\end{matrix} \,\middle|\, x\right).$$

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the finite field version of

$$ {}_{n+1}F_n\left(\begin{matrix} \frac12, \frac12, \ldots, \frac12 \\ 1, \ldots, 1 \end{matrix} \middle| x \right).$$

- Since $p^n {}_{n+1}F_n(x) \in \mathbb{Z}$, it follows easily that

$$b(p) \equiv -p^5 {}_6F_5(1) \equiv {}_6F_5 \left(\begin{matrix} \frac12, \frac12, \frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \pmod{p}.$$

A brief impression of the available ingredients, cont'd

THM Osburn Schneider 2009

For primes $p > 2$ and $\ell \geqslant 2$,

$$-p^{2\ell-1}{}_{2\ell}F_{2\ell-1}(1) \equiv p^2 X_\ell(p) + pY_\ell(p) + Z_\ell(p) \pmod{p^3}.$$

- With $m = (p-1)/2$, the right-hand sides are

$$Z_\ell(p) = {}_{2\ell}F_{2\ell-1}\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1\end{matrix}\middle| 1\right)_m,$$

A brief impression of the available ingredients, cont'd

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$$Y_\ell(p) = \sum_{k=0}^{m} (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 - \ell k(2H_k - H_{m+k} - H_{m-k}),$$

$$X_\ell(p) = \sum_{k=0}^{m} (-1)^{\ell k} \binom{m+k}{k}^\ell \binom{m}{k}^\ell (1 + 4\ell k(H_{m+k} - H_k) + 2\ell^2 k^2 (H_{m+k} - H_k)^2 - \ell k^2 (H_{m+k}^{(2)} - H_k^{(2)})).$$

A harmonic identity

THM

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \left(1 - 2k(2H_k - H_{n+k} - H_{n-k})\right) = 1$$

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- As Nesterenko (1996), consider the partial fraction decomposition

$$R(t) = \frac{\prod_{j=1}^{n}(t-j)^2}{\prod_{j=0}^{n}(t+j)^2} = \sum_{k=0}^{n} \left(\frac{A_k}{(t+k)^2} + \frac{B_k}{t+k} \right).$$

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- One finds

$$A_k = \binom{n+k}{k}^2 \binom{n}{k}^2,$$

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- The residue sum theorem applied to $tR(t)$ implies:

$$\sum_{k=0}^{n} (A_k - kB_k) = \sum_{\text{finite poles } x} \operatorname{Res}_x tR(t) = -\operatorname{Res}_\infty tR(t) = 1$$

- Only needed modulo p^2 and $n = (p-1)/2$ for Kilbourn's congruence.

A harmonic congruence

- Using identities similarly obtained from partial fractions, the ${}_6F_5$ congruence can be reduced to:

LEM
OSZ 2017

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}^3\binom{n}{k}^3\left(1-3k(2H_k-H_{n+k}-H_{n-k})\right) \equiv \sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2 \pmod{p^2}$$

for primes $p > 2$ and $n = (p-1)/2$.

- While identities can (now) be verified algorithmically, no algorithms are available for proving such congruences.

Paule–Schneider harmonic sums

DEF Paule, Schneider 2003

$$C_\ell(n) = \sum_{k=0}^{n} \binom{n}{k}^{\ell} \big(1 - \ell k (H_k - H_{n-k})\big)$$

- These are integer sequences: $C_1(n) = 1$, $C_2(n) = 0$, $C_3(n) = (-1)^n$,

$$C_4(n) = (-1)^n \binom{2n}{n}, \quad C_5(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

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LEM OSZ '17; Chu, De Donno '05

$$C_6(n) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n}$$

- Open question: are there single-sum hypergeometric expressions for $C_\ell(n)$ when $\ell \geqslant 7$?

Another Apéry supercongruence

LEM OSZ '17 For all odd primes p,

$$A\left(\frac{p-1}{2}\right) \equiv C_6\left(\frac{p-1}{2}\right) \pmod{p^2}.$$

- Modular parametrizations by weight 2 modular forms of level 6 and 7.
- In other words,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \equiv (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \pmod{p^2}.$$

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- Proving this congruence is easy once we replace the right-hand side with

$$C_6(n) = \sum_{k=0}^{n} (-1)^k \binom{3n+1}{n-k} \binom{n+k}{k}^3.$$

- Again, let us lament the lack of an algorithmic approach to such congruences.

An irrational equality

LEM

$$A(n) = \frac{(-1)^n}{2} \sum_{k=0}^{n} \binom{n+k}{n} \binom{2n-k}{n} \binom{n}{k}^4 \times \left(2 + (n-2k)(5H_k - 5H_{n-k} - H_{n+k} + H_{2n-k})\right)$$

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- This arises from a construction of linear forms in $\zeta(3)$ due to Ball. If

$$\widehat{R}(t) = \frac{n!^2\,(2t+n)\prod_{j=1}^{n}(t-j)\cdot\prod_{j=1}^{n}(t+n+j)}{\prod_{j=0}^{n}(t+j)^4} = \sum_{k=0}^{n}\left(\frac{\widehat{A}_k}{(t+k)^4} + \frac{\widehat{B}_k}{(t+k)^3} + \frac{\widehat{C}_k}{(t+k)^2} + \frac{\widehat{D}_k}{t+k}\right),$$

then $\sum_{t=1}^{\infty} \widehat{R}(t) = u_n \zeta(3) + v_n$.

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$$A(n) = \frac{(-1)^n}{2} \sum_{k=0}^{n} \binom{n+k}{n} \binom{2n-k}{n} \binom{n}{k}^4 \times \left(2 + (n-2k)(5H_k - 5H_{n-k} - H_{n+k} + H_{2n-k})\right)$$

- This arises from a construction of linear forms in $\zeta(3)$ due to Ball. If

$$\widehat{R}(t) = \frac{n!^2\,(2t+n)\prod_{j=1}^{n}(t-j)\cdot\prod_{j=1}^{n}(t+n+j)}{\prod_{j=0}^{n}(t+j)^4} = \sum_{k=0}^{n}\left(\frac{\widehat{A}_k}{(t+k)^4} + \frac{\widehat{B}_k}{(t+k)^3} + \frac{\widehat{C}_k}{(t+k)^2} + \frac{\widehat{D}_k}{t+k}\right),$$

  then $\sum_{t=1}^{\infty} \widehat{R}(t) = u_n\zeta(3) + v_n$.
- Remarkably, these linear forms agree with Apéry's:

$$A(n) = \frac{1}{2}u_n = \frac{1}{2}\sum_{k=0}^{n}\widehat{B}_k$$

Outlook

- Can we extend the congruence

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right)_{p-1} \equiv b(p) \pmod{p^3}, $$

  and show that it holds modulo p^5?

  Special relevance of p^3: by Weil's bounds, $|b(p)| < 2p^{5/2}$

- Can the algorithmic approaches for $A = B$ be adjusted to $A \equiv B$?

- Why do these supercongruences hold?

  Very promising explanation suggested by Roberts, Rodriguez-Villegas, Watkins (2017) in terms of gaps between Hodge numbers of an associated motive.

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

Robert Osburn, Armin Straub and Wadim Zudilin
A modular supercongruence for $_6F_5$: An Apéry-like story
Preprint, 2017. arXiv:1701.04098