The congruences of Fermat, Euler, Gauss and stronger versions thereof

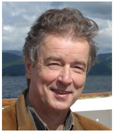
Algebra and Number Theory Seminar
LSU

Armin Straub

Oct 22, 2018

University of South Alabama

includes work with



Frits Beukers
(Utrecht University)

Marc Houben
(Utrecht University)

and

Dermot McCarthy
(Texas Tech)

Robert Osburn
(UCD)

Outline

- Introduction: Multivariate generating functions
- Gauss congruences
- Apéry-like sequences
- Multivariate supercongruences
- Brown's cellular integrals
- (Time permitting) Further open problems
- (Time permitting) Polynomial analogs

I

Introduction: MGFs

Introduction: Diagonals

- Given a series

$$F(x_1, \ldots, x_d) = \sum_{n_1, \ldots, n_d \geqslant 0} a(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n, \ldots, n)$.

EG The diagonal coefficients of

$$\frac{1}{1 - x - y} = \sum_{n=0}^{\infty} (x + y)^n$$

are the central binomial coefficients $\binom{2n}{n}$.

- Given a series

$$F(x_1,\ldots,x_d)=\sum_{n_1,\ldots,n_d\geqslant 0} a(n_1,\ldots,n_d)x_1^{n_1}\cdots x_d^{n_d},$$

its **diagonal coefficients** are the coefficients $a(n,\ldots,n)$.

EG The diagonal coefficients of

$$\frac{1}{1-x-y}=\sum_{n=0}^{\infty}(x+y)^n$$

are the central binomial coefficients $\binom{2n}{n}$.
For comparison, their univariate generating function is

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}.$$

EG The **Lucas numbers** L_n have OGF $\frac{2-x}{1-x-x^2}$.

$$L_{n+1} = L_n + L_{n-1}$$
$$L_0 = 2, L_1 = 1$$

- The sequences with rational OGF are precisely the C**-finite** ones.

EG The **Lucas numbers** L_n have OGF $\frac{2-x}{1-x-x^2}$.

$$L_{n+1} = L_n + L_{n-1}$$
$$L_0 = 2, L_1 = 1$$

- The sequences with rational OGF are precisely the **C-finite** ones.

EG The **Delannoy numbers** have OGF $\frac{1}{\sqrt{1-6x+x^2}}$.

$$D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$$

They are the diagonal of $\frac{1}{1-x-y-xy}$.

- The sequences with algebraic OGF are precisely the diagonals of 2-variable rational functions.

EG The **Lucas numbers** L_n have OGF $\frac{2-x}{1-x-x^2}$.

$L_{n+1} = L_n + L_{n-1}$
$L_0 = 2, L_1 = 1$

- The sequences with rational OGF are precisely the **C-finite** ones.

EG The **Delannoy numbers** have OGF $\frac{1}{\sqrt{1-6x+x^2}}$.

$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$

They are the diagonal of $\frac{1}{1-x-y-xy}$.

- The sequences with algebraic OGF are precisely the diagonals of 2-variable rational functions.

THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F \in K[[x_1, \ldots, x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1, \ldots, x_d)$.

EG The **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$ are the diagonal of

$$\frac{1}{1-x-y-z+4xyz}.$$

Their OGF is

$$\frac{1}{1-2x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\1\end{matrix}\right|\frac{27x^2}{(1-2x)^3}\right).$$

EG The **Franel numbers** $\sum_{k=0}^{n} \binom{n}{k}^3$ are the diagonal of

$$\frac{1}{1-x-y-z+4xyz}.$$

Their OGF is

$$\frac{1}{1-2x}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3},\frac{2}{3}\\ 1\end{matrix}\right| \frac{27x^2}{(1-2x)^3}\right).$$

- Not at all unique! The Franel numbers are also the diagonal of

$$\frac{1}{(1-x)(1-y)(1-z)-xyz}.$$

Introduction: Apéry numbers

THM
S 2014

The **Apéry numbers** are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

THM S 2014 The **Apéry numbers** are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

Introduction: Apéry numbers

THM
S 2014

The **Apéry numbers** are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\left.\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\right| -\frac{1024x}{(1-x+z)^4}\right),$$

  where $z=\sqrt{1-34x+x^2}$.

- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson

THM S 2014 The **Apéry numbers** are the diagonal coefficients of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix}\middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- OGFs of such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod 8, & \text{if } n \text{ even},\\ 5 \pmod 8, & \text{if } n \text{ odd}.\end{cases}$$

Chowla–Cowles–Cowles '80
Rowland–Yassawi '13

II

Gauss congruences

Fermat, Euler and Gauss congruences

DEF $a(n)$ satisfies the **Fermat congruences** if, for all primes p,

$$a(p) \equiv a(1) \pmod{p}.$$

EG Classical: $a(n) = a^n$ satisfies the Fermat congruences.

Fermat, Euler and Gauss congruences

DEF $a(n)$ satisfies the **Fermat congruences** if, for all primes p,

$$a(p) \equiv a(1) \pmod{p}.$$

EG Classical: $a(n) = a^n$ satisfies the Fermat congruences.

In fact, we know that these sequences satisfy stronger congruences:

DEF $a(n)$ satisfies the **Euler congruences** if, for all primes p,

$$a(p^r) \equiv a(p^{r-1}) \pmod{p^r}.$$

Fermat, Euler and Gauss congruences

DEF $a(n)$ satisfies the **Fermat congruences** if, for all primes p,

$$a(p) \equiv a(1) \pmod{p}.$$

EG Classical: $a(n) = a^n$ satisfies the Fermat congruences.

In fact, we know that these sequences satisfy stronger congruences:

DEF $a(n)$ satisfies the **Euler congruences** if, for all primes p,

$$a(p^r) \equiv a(p^{r-1}) \pmod{p^r}.$$

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

Equivalently, $\displaystyle\sum_{d|m} \mu(\tfrac{m}{d}) a(d) \equiv 0 \pmod{m}$.

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

EG

- $a(n) = a^n$

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $\begin{array}{c} L_{n+1} = L_n + L_{n-1} \\ L_0 = 2, L_1 = 1 \end{array}$

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $\begin{matrix} L_{n+1} = L_n + L_{n-1} \\ L_0 = 2, L_1 = 1 \end{matrix}$
- $a(n) = D_n$ **Delannoy numbers**: $D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$

Gauss congruences

DEF $a(n)$ satisfies the **Gauss congruences** if, for all primes p,

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r}.$$

EG

- $a(n) = a^n$
- $a(n) = L_n$ **Lucas numbers**: $L_{n+1} = L_n + L_{n-1}$, $L_0 = 2, L_1 = 1$
- $a(n) = D_n$ **Delannoy numbers**: $D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$

- Later, we allow $a(n) \in \mathbb{Q}$. If the Gauss congruences hold for all but finitely many p, we say that the sequence (or its GF) has the **Gauss property**.
- Similarly, for multivariate sequences $a(\boldsymbol{n})$, we require

$$a(\boldsymbol{m}p^r) \equiv a(\boldsymbol{m}p^{r-1}) \pmod{p^r}.$$

That is, for instance, for $a(n_1, n_2)$,

$$a(m_1 p^r, m_2 p^r) \equiv a(m_1 p^{r-1}, m_2 p^{r-1}) \pmod{p^r}.$$

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

In fact, up to a positivity condition, (G) characterizes realizability.

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

  $$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

  Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

  In fact, up to a positivity condition, (G) characterizes realizability.

- $a(n) = \operatorname{trace}(M^n)$ — Jänichen '21, Schur '37; also: Arnold, Zarelua

  where M is an integer matrix

More sequences satisfying Gauss congruences

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

  $$a(n) = \#\{x \in X : T^n x = x\} \qquad \text{“points of period } n\text{”}$$

  Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

  In fact, up to a positivity condition, (G) characterizes realizability.

- $a(n) = \operatorname{trace}(M^n)$ — Jänichen '21, Schur '37; also: Arnold, Zarelua

  where M is an integer matrix

- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.

Minton's theorem

THM Minton, 2014: $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

Minton's theorem

THM (Minton, 2014) $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^{s}(1 - \alpha_i x)$ then

$$x\frac{u'(x)}{u(x)} = -\sum_{i=1}^{s}\frac{\alpha_i x}{1 - \alpha_i x} = s - \sum_{i=1}^{s}\frac{1}{1 - \alpha_i x}.$$

THM Minton, 2014: $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^{s}(1-\alpha_i x)$ then

$$x\frac{u'(x)}{u(x)} = -\sum_{i=1}^{s}\frac{\alpha_i x}{1-\alpha_i x} = s - \sum_{i=1}^{s}\frac{1}{1-\alpha_i x}.$$

- Assuming the α_i are distinct,

$$\sum_{i=1}^{s}\frac{1}{1-\alpha_i x} = \sum_{n\geqslant 0}\left(\sum_{i=1}^{s}\alpha_i^n\right)x^n = \sum_{n\geqslant 0}\operatorname{trace}(M^n)x^n,$$

where M is the companion matrix of $\prod_{i=1}^{s}(x-\alpha_i) = x^s u(1/x)$.

Minton's theorem

THM Minton, 2014: $f \in \mathbb{Q}(x)$ has the Gauss property if and only if f is a $\mathbb{Q}$-linear combination of functions $xu'(x)/u(x)$, with $u \in \mathbb{Z}[x]$.

- If $u(x) = \prod_{i=1}^s (1 - \alpha_i x)$ then

$$x \frac{u'(x)}{u(x)} = -\sum_{i=1}^s \frac{\alpha_i x}{1 - \alpha_i x} = s - \sum_{i=1}^s \frac{1}{1 - \alpha_i x}.$$

- Assuming the α_i are distinct,

$$\sum_{i=1}^s \frac{1}{1 - \alpha_i x} = \sum_{n \geqslant 0} \left(\sum_{i=1}^s \alpha_i^n \right) x^n = \sum_{n \geqslant 0} \mathrm{trace}(M^n) x^n,$$

where M is the companion matrix of $\prod_{i=1}^s (x - \alpha_i) = x^s u(1/x)$.

- Minton: No new C-finite sequences with the Gauss property!
- Can we generalize from C-finite towards D-finite?

The multivariate case

THM BHS
Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

Here, $N(Q)$ is the Newton polytope of Q. In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

THM BHS: Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.
Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

Here, $N(Q)$ is the Newton polytope of Q. In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

EG Beukers, Houben, S 2017: The **Delannoy numbers** $D(n_1, n_2)$ are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D(n_1,n_2) x^{n_1} y^{n_2}.$$

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

Here, $N(Q)$ is the Newton polytope of Q. In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

EG Beukers, Houben, S 2017

The **Delannoy numbers** $D(n_1, n_2)$ are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D(n_1,n_2) x^{n_1} y^{n_2}.$$

By the theorem, the following have the Gauss property:

$$\frac{N}{1-x-y-xy} \quad \text{with } N \in \{1, x, y, xy\}$$

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

Here, $N(Q)$ is the Newton polytope of Q. In this case, $N(Q) = \operatorname{supp}(Q) \subseteq \{0,1\}^n$.

EG Beukers, Houben, S 2017

The **Delannoy numbers** $D(n_1, n_2)$ are characterized by

$$\frac{1}{1-x-y-xy} = \sum_{n_1,n_2=0}^{\infty} D(n_1,n_2) x^{n_1} y^{n_2}.$$

By the theorem, the following have the Gauss property:

$$\frac{N}{1-x-y-xy} \quad \text{with } N \in \{1, x, y, xy\}$$

In other words, for $\boldsymbol{\delta} \in \{0,1\}^2$,

$$D(\boldsymbol{m}p^r - \boldsymbol{\delta}) \equiv D(\boldsymbol{m}p^{r-1} - \boldsymbol{\delta}) \pmod{p^r}.$$

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{x}{Q} \frac{\partial Q}{\partial x} = \frac{-x + 4xyz}{Q}$$

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{x}{Q} \frac{\partial Q}{\partial x} = \frac{-x + 4xyz}{Q}$$

$$f_1 = Q, \quad f_2 = 1 - 4yz \quad \Longrightarrow \quad \text{(D)} = \frac{xy}{f_1 f_2} \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \frac{4xyz}{Q}$$

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Interesting detail: true for any of the different Laurent expansions of multivariate rational functions

EG Consider $Q = 1 - x - y - z + 4xyz$:

$$f_1 = Q \quad \Longrightarrow \quad \text{(D)} = \frac{x}{Q} \frac{\partial Q}{\partial x} = \frac{-x + 4xyz}{Q}$$

$$f_1 = Q, \quad f_2 = 1 - 4yz \quad \Longrightarrow \quad \text{(D)} = \frac{xy}{f_1 f_2} \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \frac{4xyz}{Q}$$

In particular, $\dfrac{1}{1 - x - y - z + 4xyz}$ has the Gauss property.

There is nothing special about 4 in this argument.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

THM BHS

Let $P, Q \in \mathbb{Z}[\boldsymbol{x}]$ with Q linear in each variable.

Then P/Q has the Gauss property if and only if $N(P) \subseteq N(Q)$.

THM BHS

Let $P, Q \in \mathbb{Z}[z, \boldsymbol{x}]$ with Q linear in $x_1, \ldots, x_n$.

Write $P = \sum_{\boldsymbol{k}} p_{\boldsymbol{k}}(z) \boldsymbol{x}^{\boldsymbol{k}}$ and $Q = \sum_{\boldsymbol{k}} q_{\boldsymbol{k}}(z) \boldsymbol{x}^{\boldsymbol{k}}$.

Then P/Q has the Gauss property if and only if

- $p_{\boldsymbol{k}} \neq 0$ implies $q_{\boldsymbol{k}} \neq 0$ and
- $p_{\boldsymbol{k}}/q_{\boldsymbol{k}}$ has the Gauss property whenever $q_{\boldsymbol{k}} \neq 0$.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \tag{D}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.
- Yes, for $f = P/Q$ with Q in two variables and total degree 2.

The multivariate case, cont'd

THM Beukers, Houben, S 2017

Let $f_1, \ldots, f_m \in \mathbb{Q}(\boldsymbol{x}) = \mathbb{Q}(x_1, \ldots, x_n)$ be nonzero. Then

$$\frac{x_1 \cdots x_m}{f_1 \cdots f_m} \det \left(\frac{\partial f_j}{\partial x_i} \right)_{i,j=1,\ldots,m} \qquad \text{(D)}$$

has the Gauss property.

Q BHS

Suppose $f \in \mathbb{Q}(\boldsymbol{x})$ has the Gauss property. Can it be written as a $\mathbb{Q}$-linear combination of functions of the form (D)?

- Yes, for $n = 1$, by Minton's theorem.
- Yes, for $f = P/Q$ with Q linear in all, or all but one, variables.
- Yes, for $f = P/Q$ with Q in two variables and total degree 2.

EG Can $\dfrac{x(x+y+y^2+2xy^2)}{1+3x+3y+2x^2+2y^2+xy-2x^2y^2}$ be written in that form?

III

Apéry-like sequences

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a,b,c) = (17,5,1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q (Beukers, Zagier) Are there other tuples (a,b,c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a,b,c)=(17,5,1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - cn^3 u_{n-1}.$$

Q (Beukers, Zagier): Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

  with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha=2^4,3^3,2^6,2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}} .$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)} \right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $q - 12q^2 + 66q^3 + O(q^4)$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

Supercongruences for Apéry numbers

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,
$$A(p) \equiv 5 \pmod{p^3}.$$
- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$
$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88: The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

EG For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

- Chowla, Cowles, Cowles (1980) conjectured that, for primes $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

EG

Mathematica 7 miscomputes $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ for $n > 5500$.

$$A(5 \cdot 11^3) = 12488301 \ldots \text{about 2000 digits} \ldots \text{about 8000 digits} \ldots 7956522125$$

Weirdly, with this wrong value, one still has

$$A(5 \cdot 11^3) \equiv A(5 \cdot 11^2) \pmod{11^6}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Beukers, Coster '87-'88
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '16
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

CONJ
Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

$$s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}$$

Cooper's sporadic sequences

- Cooper's search for integral solutions to

$$(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$$

revealed three additional sporadic solutions: s_{10} and supercongruence known

$$s_7(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \binom{2k}{n} \qquad s_{10}(n) = \sum_{k=0}^{n} \binom{n}{k}^4$$

$$s_{18}(n) = \sum_{k=0}^{[n/3]} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \left[\binom{2n-3k-1}{n} + \binom{2n-3k}{n} \right]$$

CONJ Cooper 2012

$$s_7(mp) \equiv s_7(m) \pmod{p^3} \qquad p \geqslant 3$$

$$s_{18}(mp) \equiv s_{18}(m) \pmod{p^2}$$

THM Osburn-Sahu-S 2016

$$s_7(mp^r) \equiv s_7(mp^{r-1}) \pmod{p^{3r}} \qquad p \geqslant 5$$

$$s_{18}(mp^r) \equiv s_{18}(mp^{r-1}) \pmod{p^{2r}}$$

IV

Multivariate supercongruences

Multivariate supercongruences

THM
S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

THM S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n} p^r) \equiv A(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

- $\sum_{n \geqslant 0} a(n) x^n = F(x) \quad \Longrightarrow \quad \sum_{n \geqslant 0} a(pn) x^{pn} = \frac{1}{p} \sum_{k=0}^{p-1} F(\zeta_p^k x) \qquad \zeta_p = e^{2\pi i/p}$
- Hence, both $A(\boldsymbol{n} p^r)$ and $A(\boldsymbol{n} p^{r-1})$ have rational generating function.

  The proof, however, relies on an explicit binomial sum for the coefficients.

THM S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}.$$

THM S 2014

Define $A(\boldsymbol{n}) = A(n_1, n_2, n_3, n_4)$ by

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} A(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}.$$

- The Apéry numbers are the diagonal coefficients.
- For $p \geqslant 5$, we have the **multivariate supercongruences**

$$A(\boldsymbol{n}p^r) \equiv A(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- By MacMahon's Master Theorem,

$$A(\boldsymbol{n}) = \sum_{k\in\mathbb{Z}} \binom{n_1}{k}\binom{n_3}{k}\binom{n_1+n_2-k}{n_1}\binom{n_3+n_4-k}{n_3}.$$

- Because $A(n-1) = A(-n,-n,-n,-n)$, we also find

$$A(mp^r-1) \equiv A(mp^{r-1}-1) \pmod{p^{3r}}.$$

Beukers '85

More conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

  $1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$

$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$

$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$

$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$

$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$

$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

More conjectural multivariate supercongruences

- Exhaustive search by Alin Bostan and Bruno Salvy:

  $1/(1-p(x,y,z,w))$ with $p(x,y,z,w)$ a sum of distinct monomials; Apéry numbers as diagonal

$$\frac{1}{1-(x+y+xy)(z+w+zw)}$$

$$\frac{1}{1-(1+w)(z+xy+yz+zx+xyz)}$$

$$\frac{1}{1-(y+z+xy+xz+zw+xyw+xyzw)}$$

$$\frac{1}{1-(y+z+xz+wz+xyw+xzw+xyzw)}$$

$$\frac{1}{1-(z+xy+yz+xw+xyw+yzw+xyzw)}$$

$$\frac{1}{1-(z+(x+y)(z+w)+xyz+xyzw)}$$

CONJ
S 2014

The coefficients $B(\boldsymbol{n})$ of each of these satisfy, for $p \geqslant 5$,

$$B(\boldsymbol{n}p^r) \equiv B(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

An infinite family of rational functions

THM S 2014

Let $\lambda \in \mathbb{Z}_{>0}^{\ell}$ with $d = \lambda_1 + \ldots + \lambda_\ell$. Define $A_\lambda(\boldsymbol{n})$ by

$$\frac{1}{\prod_{1 \leqslant j \leqslant \ell} \left[1 - \sum_{1 \leqslant r \leqslant \lambda_j} x_{\lambda_1 + \ldots + \lambda_{j-1} + r}\right] - x_1 x_2 \cdots x_d} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^d} A_\lambda(\boldsymbol{n}) \boldsymbol{x}^{\boldsymbol{n}}.$$

- If $\ell \geqslant 2$, then, for all primes p,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{2r}}.$$

- If $\ell \geqslant 2$ and $\max(\lambda_1, \ldots, \lambda_\ell) \leqslant 2$, then, for primes $p \geqslant 5$,

$$A_\lambda(\boldsymbol{n} p^r) \equiv A_\lambda(\boldsymbol{n} p^{r-1}) \pmod{p^{3r}}.$$

EG

$\lambda = (2,2)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}$$

$\lambda = (2,1)$

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3) - x_1 x_2 x_3}$$

Further examples

EG

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}$$

has as diagonal the $\zeta(2)$ Apéry numbers

$$B(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

EG

$$\frac{1}{(1-x_1)(1-x_2)\cdots(1-x_d)-x_1x_2\cdots x_d}$$

has as diagonal the numbers — $d=3$: Franel, $d=4$: Yang–Zudilin

$$Y_d(n)=\sum_{k=0}^{n}\binom{n}{k}^d.$$

- In each case, we obtain supercongruences generalizing results of Coster (1988) and Chan–Cooper–Sica (2010).

A conjectural multivariate supercongruence

CONJ S 2014 The coefficients $Z(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^4} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- Here, the diagonal coefficients are the **Almkvist–Zudilin numbers**

$$Z(n) = \sum_{k=0}^{n} (-3)^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3},$$

for which the univariate congruences are still open for $r > 1$.

V

Brown's cellular integrals

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)^{n+1}}\, dxdy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\, dxdydw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dx dy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}}\, dx dy dw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

where $\tilde{a}(n), \tilde{b}(n) \in \mathbb{Q}$ and

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dxdy$$
$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}}\, dxdydw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

where $\tilde{a}(n), \tilde{b}(n) \in \mathbb{Q}$ and

$$a(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad b(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

- Brown realizes these as period integrals, for $N = 5, 6$, on the moduli space $\mathcal{M}_{0,N}$ of curves of genus 0 with N marked points.

Brown's cellular integrals

THM Brown 2009: Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i} (1-t_j)^{b_j} (t_i - t_j)^{c_{ij}} dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.

Brown's cellular integrals

THM Brown 2009: Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i}(1-t_j)^{b_j}(t_i-t_j)^{c_{ij}} dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.
- Brown constructs families of integrals $I_\sigma(n)$, for which MZVs of submaximal weight vanish.

  Here, σ are certain ("convergent") permutations in S_N.

N	5	6	7	8	9	10	11
# of σ	1	1	5	17	105	771	7028

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG **Panzer: HyperInt**

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \tfrac{515189}{6}\zeta(2) - \tfrac{331063}{48}$$

- OGF of $I_\sigma(n)$ satisfies a Picard–Fuchs DE of order 7 (Lairez). With 2-dimensional space of analytic solutions at 0.

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \tfrac{515189}{6}\zeta(2) - \tfrac{331063}{48}$$

- OGF of $I_\sigma(n)$ satisfies a Picard–Fuchs DE of order 7 (Lairez).
  With 2-dimensional space of analytic solutions at 0.
- The leading coefficients of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

CONJ McCarthy, Osburn, S 2018 For each $N \geqslant 5$ and convergent σ_N, the leading coefficients $A_{\sigma_N}(n)$ satisfy $(p \geqslant 5)$

$$A_{\sigma_N}(mp^r) \equiv A_{\sigma_N}(mp^{r-1}) \pmod{p^{3r}}.$$

For $N = 5, 6$ these are the supercongruences proved by Beukers and Coster.

One of Brown's cellular integrals, cont'd

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where
- The leading coefficients $A_\sigma(n)$ of $I_\sigma(n)$ are:

$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i} \binom{n+k_i}{k_i}$$

THM McCarthy, Osburn, S 2018

For any odd prime p,

$$A_\sigma\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p^2}.$$

where $\eta^{12}(2z) = \sum_{n \geqslant 1} \gamma(n) q^n$ is the unique newform in $S_6(\Gamma_0(4))$.

The Ahlgren–Ono supercongruences

THM Ahlgren–Ono '00

For any odd prime p, the $\zeta(3)$ Apéry numbers satisfy

$$A\left(\frac{p-1}{2}\right) \equiv \alpha(p) \pmod{p^2},$$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

THM Ahlgren '01

For any prime $p \geqslant 5$, the $\zeta(2)$ Apéry numbers satisfy

$$B\left(\frac{p-1}{2}\right) \equiv \beta(p) \pmod{p^2},$$

with $\eta(4\tau)^6 = \sum_{n\geqslant 1} \beta(n) q^n$ the unique newform in $S_3(\Gamma_0(16), (\frac{-4}{\cdot}))$.

- conjectured (and proved modulo p) by Beukers '87

- $A_{\sigma_N}(n) = B(n)^{(N-3)/2}$ is one of Brown's sequences for a certain σ_N.
  Here, $B(n)$ are the $\zeta(2)$ Apéry numbers.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2}(n - im)^{k-1} q^{n^2+m^2} = \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, Osburn, S 2018

Let $N \geqslant 5$ be odd. For any prime $p \geqslant 5$,

$$A_{\sigma_N}\left(\frac{p-1}{2}\right) \equiv \gamma_{N-2}(p) \pmod{p^2}.$$

An infinite family of supercongruences

- $A_{\sigma_N}(n) = B(n)^{(N-3)/2}$ is one of Brown's sequences for a certain σ_N.
  Here, $B(n)$ are the $\zeta(2)$ Apéry numbers.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2}(n-im)^{k-1} q^{n^2+m^2} = \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, Osburn, S 2018

Let $N \geqslant 5$ be odd. For any prime $p \geqslant 5$,

$$A_{\sigma_N}\left(\frac{p-1}{2}\right) \equiv \gamma_{N-2}(p) \pmod{p^2}.$$

Q Supercongruences for all of Brown's sequences?
Maybe arising from L-series attached to Galois representations?

Hypergeometric supercongruences

THM Kilbourn 2006

$(p \geqslant 3)$

$$ {}_4F_3 \left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv \alpha(p) \pmod{p^3}, $$

with $\eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

Hypergeometric supercongruences

THM
Kilbourn 2006

$(p \geqslant 3)$

$$ {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv \alpha(p) \pmod{p^3}, $$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series ${}_4F_3$ and
  - Fourier coefficients of modular forms of weight 4.
- 11 of these remained open until very recently proved by Long, Tu, Yui, Zudilin (2017).

McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves "half" of each of the 14.

THM Kilbourn 2006

$$ {}_4F_3\left(\begin{matrix}\frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv \alpha(p) \pmod{p^3}, \qquad (p \geqslant 3)$$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} \alpha(n)q^n$ the unique newform in $S_4(\Gamma_0(8))$.

- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series ${}_4F_3$ and
  - Fourier coefficients of modular forms of weight 4.
- 11 of these remained open until very recently proved by Long, Tu, Yui, Zudilin (2017).

McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves "half" of each of the 14.

Q Can the supercongruences for Brown's sequences be similarly embedded in the hypergeometric setting?

THM Osburn, S, Zudilin 2018

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1\right)_{p-1} \equiv \lambda(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $\lambda(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n\geqslant 1} \lambda(n)q^n \in S_6(\Gamma_0(8)). $$

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.

A supercongruence for ${}_6F_5$

THM Osburn, S, Zudilin 2018

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix}\middle| 1\right)_{p-1} \equiv \lambda(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $\lambda(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n \geqslant 1} \lambda(n) q^n \in S_6(\Gamma_0(8)). $$

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.
- A result of Frechette, Ono and Papanikolas expresses the $\lambda(p)$ in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo p^3 in terms of sums involving harmonic sums.

A supercongruence for ${}_6F_5$

THM Osburn, S, Zudilin 2018

$$ {}_6F_5\left(\begin{matrix} \frac12, \frac12, \frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv \lambda(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $\lambda(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8\eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n\geqslant 1} \lambda(n) q^n \in S_6(\Gamma_0(8)). $$

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.
- A result of Frechette, Ono and Papanikolas expresses the $\lambda(p)$ in terms of Gaussian hypergeometric functions.
- Osburn and Schneider determined the resulting Gaussian hypergeometric functions modulo p^3 in terms of sums involving harmonic sums.

Q Why do these supercongruences hold?

Promising explanation suggested by Roberts, Rodriguez-Villegas (2017) in terms of gaps between Hodge numbers of an associated motive.

VI

Further open problems

Irrationality and transcendence of zeta values

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Open: $\zeta(5)$ is irrational

Irrationality and transcendence of zeta values

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental

Irrationality and transcendence of zeta values

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational

Irrationality and transcendence of zeta values

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational

Binomial supercongruences modulo more than p^3?

- Are there primes p (**Wolstenholme primes**) such that

$$\binom{2p}{p} \equiv 2 \pmod{p^4}?$$

Equivalently, $H_{p-1} \equiv 0 \pmod{p^3}$. Or, $B_{p-3} \equiv 0 \pmod{p}$.

- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Binomial supercongruences modulo more than p^3?

- Are there primes p (**Wolstenholme primes**) such that

$$\binom{2p}{p} \equiv 2 \pmod{p^4}?$$

  Equivalently, $H_{p-1} \equiv 0 \pmod{p^3}$. Or, $B_{p-3} \equiv 0 \pmod{p}$.

- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9
- Infinitely many Wolstenholme primes are conjectured to exist.

  Namely, about $\log(\log(x))$ many up to x.

  However, no primes are conjectured to exist for modulo p^5.

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

Binomial supercongruences modulo more than p^3?

- Are there primes p (**Wolstenholme primes**) such that

$$\binom{2p}{p} \equiv 2 \pmod{p^4}?$$

Equivalently, $H_{p-1} \equiv 0 \pmod{p^3}$. Or, $B_{p-3} \equiv 0 \pmod{p}$.

- The only two known are 16843 and 2124679. McIntosh, 1995: up to 10^9
- Infinitely many Wolstenholme primes are conjectured to exist.

Namely, about $\log(\log(x))$ many up to x.

However, no primes are conjectured to exist for modulo p^5.

CONJ

$$\binom{2n}{n} \equiv 2 \pmod{n^3} \quad \Longleftrightarrow \quad n \geqslant 5 \text{ and } n \text{ is prime}$$

This would imply (Jones, '94) that there exists a polynomial in 7 variables, whose positive range is exactly the prime numbers. (Known: 10 variables (Matijasevic, '77))

C. Helou and G. Terjanian
On Wolstenholme's theorem and its converse
Journal of Number Theory 128, 2008

A simple multivariate supercongruence

CONJ S 2014 The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- The diagonal coefficients are the **Franel numbers** $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$.

A simple multivariate supercongruence

CONJ S 2014 The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_{\geqslant 0}} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- The diagonal coefficients are the **Franel numbers** $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$.
- The Franel numbers also are the diagonal coefficients of

$$\frac{1}{(1-x_1)(1-x_2)(1-x_3)-x_1x_2x_3},$$

for which the above multivariate supercongruences are known (S '14).

A simple multivariate supercongruence

CONJ S 2014 The coefficients $F(\boldsymbol{n})$ of

$$\frac{1}{1-(x_1+x_2+x_3)+4x_1x_2x_3} = \sum_{\boldsymbol{n}\in\mathbb{Z}_{\geqslant 0}^3} F(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

satisfy, for $p \geqslant 5$, the multivariate supercongruences

$$F(\boldsymbol{n}p^r) \equiv F(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

- This is a "warm-up" case for

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4},$$

  which has the Almkvist–Zudilin numbers as diagonal coefficients, for which even the univariate supercongruences remain open for $r > 1$.
- No supercongruences for the extension to more than 4 variables.

Diagonals of rational functions

THM The diagonal of a rational function is D-finite.

CONJ (Christol '90) If an integer sequence of at most exponential growth is D-finite, then it is the diagonal of a rational function.

Diagonals of rational functions

THM The diagonal of a rational function is D-finite.

CONJ Christol '90 If an integer sequence of at most exponential growth is D-finite, then it is the diagonal of a rational function.

- Bostan, Lairez and Salvy (2017) show that diagonals of rational functions are exactly **binomial sums**.
- Furstenberg (1967) shows that diagonals of rational functions have algebraic OGF modulo (almost all) primes p.

A. Bostan, P. Lairez, B. Salvy
Multiple binomial sums
Journal of Symbolic Computation 80, 2017

Rational functions in more than two variables

- Apéry numbers for $\zeta(2)$ are diagonal of

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}.$$

- Apéry numbers for $\zeta(3)$ are diagonal of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

(And we have seen several other MGFs in 4 variables.)

Q Is it possible to find a rational MGF for the $\zeta(3)$ Apéry numbers with 3 variables?

Rational functions in more than two variables

- Apéry numbers for $\zeta(2)$ are diagonal of

$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}.$$

- Apéry numbers for $\zeta(3)$ are diagonal of

$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$

(And we have seen several other MGFs in 4 variables.)

Q Is it possible to find a rational MGF for the $\zeta(3)$ Apéry numbers with 3 variables?

- diagonals of n-variable rational functions:
  - $n=1$: C-finite sequences
  - $n=2$: sequences with algebraic OGF
  - $n=3$: ?

Rational functions in more than two variables

- Apéry numbers for $\zeta(2)$ are diagonal of
$$\frac{1}{(1-x_1-x_2)(1-x_3)-x_1x_2x_3}.$$
- Apéry numbers for $\zeta(3)$ are diagonal of
$$\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}.$$
(And we have seen several other MGFs in 4 variables.)

Q Is it possible to find a rational MGF for the $\zeta(3)$ Apéry numbers with 3 variables?

- diagonals of n-variable rational functions:
  - $n=1$: C-finite sequences
  - $n=2$: sequences with algebraic OGF
  - $n=3$: ?

Q Find a sequence which is a diagonal for $n>3$ but not for $n=3$!?

VII

Polynomial analogs

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

- The q-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \binom{n}{n-k}_q$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q=1$
  and hence invisible in the classical world

The coefficients of q-binomial coefficients

- Here's some q-binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q = {} & q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ & + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ & + 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal. Sylvester, 1878

A few faces of the q-binomial coefficient

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,
- counts k-subsets of an n-set weighted by their sum,

A few faces of the q-binomial coefficient

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,
- counts k-subsets of an n-set weighted by their sum,
- features in a binomial theorem for noncommuting variables,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \qquad \text{if } yx = qxy,$$

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,
- counts k-subsets of an n-set weighted by their sum,
- features in a binomial theorem for noncommuting variables,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \qquad \text{if } yx = qxy,$$

- has a q-integral representation analogous to the beta function,

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,
- counts k-subsets of an n-set weighted by their sum,
- features in a binomial theorem for noncommuting variables,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \qquad \text{if } yx = qxy,$$

- has a q-integral representation analogous to the beta function,
- counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$.

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$
$$\equiv q^{n^2} + 1 = [2]_{q^{n^2}} \qquad \pmod{\Phi_n(q)^2}$$

(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k = 0$ or $k = n$.)

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$
$$\equiv q^{n^2} + 1 = [2]_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k = 0$ or $k = n$.)

THM Clark 1995

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

- Note that $\Phi_n(1) = 1$ if n is not a prime power.

A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$
$$\equiv q^{n^2} + 1 = [2]_{q^{n^2}} \qquad \pmod{\Phi_n(q)^2}$$

(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k = 0$ or $k = n$.)

THM Clark 1995
$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \qquad \pmod{\Phi_n(q)^2}$$

- Note that $\Phi_n(1) = 1$ if n is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \qquad \pmod{[p]_q^2}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

EG $n=13$, $a=2$, $b=1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM
S
2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

EG
$n = 13$,
$a = 2$,
$b = 1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6) = 1$.

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM
S
2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

EG
$n=13$, $a=2$, $b=1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- Ljunggren's classical congruence holds modulo p^{3+r} with r the p-adic valuation of

Jacobsthal '52

$$(a-b)ab\binom{a}{b}.$$

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

- The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

THM
S
2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$, $A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m - 1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

THM
S
2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$,

$A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m-1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.

THM
S
2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$, $\quad A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m - 1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.
- q-analog and congruences for Almkvist–Zudilin numbers? (classical supercongruences still open)

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

- **F. Beukers, M. Houben, A. Straub**
  Gauss congruences for rational functions in several variables
  Acta Arithmetica, 2018.
- **D. McCarthy, R. Osburn, A. Straub**
  Sequences, modular forms and cellular integrals
  Mathematical Proceedings of the Cambridge Philosophical Society, 2018.
- **R. Osburn, B. Sahu, A. Straub**
  Supercongruences for sporadic sequences
  Proceedings of the Edinburgh Mathematical Society, Vol. 59, Nr. 2, 2016, p. 503-518
- **A. Straub**
  Multivariate Apéry numbers and supercongruences of rational functions
  Algebra & Number Theory, Vol. 8, Nr. 8, 2014, p. 1985-2008
- **A. Straub**
  Supercongruences for polynomial analogs of the Apéry numbers
  Proceedings of the American Mathematical Society, 2018