On the Gaussian binomial coefficients, the simplest of q-series

Analytic and Combinatorial Number Theory: The Legacy of Ramanujan

A Conference in Honor of Bruce C. Berndt's 80th Birthday

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includes joint work with:

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BCB+1 day, 2017

Basic q-analogs

A q-analog reduces to the classical object in the limit $q \to 1$.

DEF

- q-number: $[n]_q = \dfrac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$

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- q-factorial: $$[n]_q! = [n]_q [n-1]_q \cdots [1]_q = \frac{(q;q)_n}{(1-q)^n}$$

For q-series fans:

D1

- q-binomial: $$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

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EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= (1-q+q^2) \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

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$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

$\Phi_6(1) = 1$ becomes invisible

Cyclotomic polynomials

DEF The nth cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

irreducible polynomial (nontrivial; Gauss!) with **integer** coefficients

- $[n]_q = \frac{q^n - 1}{q - 1} = \prod_{\substack{1 < d \leqslant n \\ d|n}} \Phi_d(q)$ For primes: $[p]_q = \Phi_p(q)$

EG

$$\Phi_5(q) = q^4 + q^3 + q^2 + q + 1$$
$$\Phi_{21}(q) = q^{12} - q^{11} + q^9 - q^8 + q^6 - q^4 + q^3 - q + 1$$
$$\begin{aligned}\Phi_{105}(q) = {} & q^{48} + q^{47} + q^{46} - q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} \\ & + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} \\ & - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^9 \\ & - q^8 - 2q^7 - q^6 - q^5 + q^2 + q + 1\end{aligned}$$

q-binomials: factored and expanded

LEM factored

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}$$

$\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor \in \{0,1\}$

proof

$$[n]_q! = \prod_{m=1}^{n} \prod_{\substack{d|m \\ d>1}} \Phi_d(q) = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor}$$

□

- In particular, the q-binomial is a polynomial. (of degree $k(n-k)$)

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EG expanded

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q &= q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ &\quad + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ &\quad + 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The coefficients are positive and **unimodal**. Sylvester, 1878

q-binomials: combinatorial

D2

THM

"normalized sum of Y"

$$\binom{n}{k}_q = \sum_Y q^{w(Y)} \quad \text{where } w(Y) = \sum_j y_j - j$$

The sum is over all k-element subsets Y of $\{1, 2, \dots, n\}$.

EG

$$\underbrace{\{1,2\}}_{\to 0}, \quad \underbrace{\{1,3\}}_{\to 1}, \quad \underbrace{\{1,4\}}_{\to 2}, \quad \underbrace{\{2,3\}}_{\to 2}, \quad \underbrace{\{2,4\}}_{\to 3}, \quad \underbrace{\{3,4\}}_{\to 4}$$

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- k-element subsets of n whose normalized sum is m,
- partitions λ of m whose Ferrer's diagram fits in a $k \times (n-k)$ box.

q-binomials: three characterizations

THM The q-binomial satisfies the **q-Pascal rule**:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D3

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D4

THM Suppose $yx = qxy$ (and that q commutes with x, y). Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j}_q x^j y^{n-j}$$

D5

DEF The q-**derivative**:

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

$$D_q x^n = \frac{(qx)^n - x^n}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q \, x^{n-1}$$

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- The q-**integral**: from formally inverting D_q

$$\int_0^x f(x)\, \mathrm{d}_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

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$$\int_0^x f(x) \, \mathrm{d}_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

- The q-**gamma function**:

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} \, \mathrm{d}_q x$$

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

D6

Can similarly define q-**beta** via a q-Euler integral.

Summary: the q-binomial coefficient

The q-binomial coefficient has a variety of natural characterizations:

- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$
- Via a q-version of **Pascal's rule**
- **Combinatorially**, as the generating function of the element sums of k-subsets of an n-set
- **Geometrically**, as the number of k-dimensional subspaces of $\mathbb{F}_q^n$
- **Algebraically**, via a binomial theorem for noncommuting variables
- **Analytically**, via q-integral representations
- Not touched here: **quantum groups** arising in representation theory and physics

Binomial coefficients with integer entries

$$\binom{-3}{5} = -21, \qquad \binom{-3}{-5} = 6$$

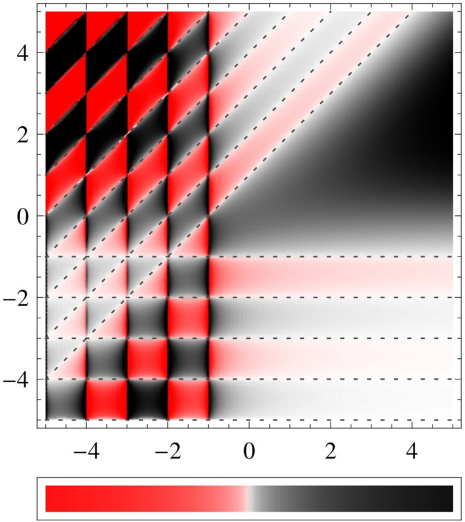
$$\binom{-3.001}{-5.001} \approx 6.004$$

$$\binom{-3.003}{-5.005} \approx 10.03$$

1989: Ph.D. at MIT (Rota)
1996+: in mathematical finance

Daniel E. Loeb
Sets with a negative number of elements
Advances in Mathematics, Vol. 91, p.64–74, 1992

A function in two variables



This scale is also visible along the line $y = 1$.

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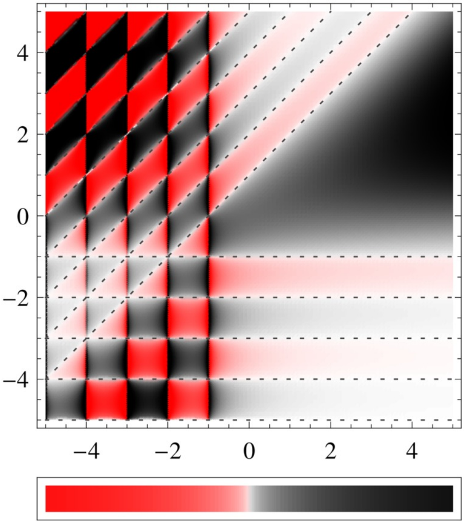

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This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R}\backslash\{x = -1, -2, \ldots\}$.

> … no evidence that the graph of C has ever been plotted before …
> **David Fowler**, *American Mathematical Monthly, Jan 1996*

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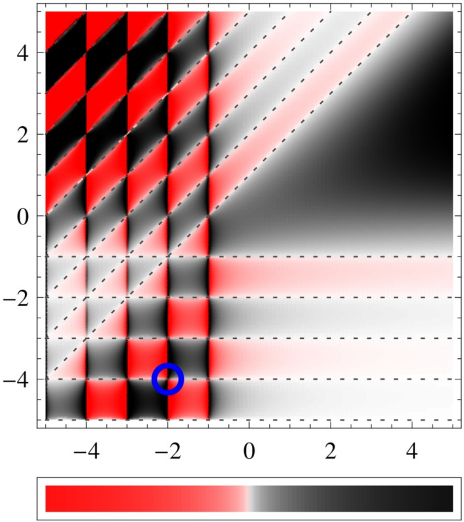

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Directional limits exist at integer points:

$$\lim_{\varepsilon\to 0} \binom{-2+\varepsilon}{-4+r\varepsilon} = \frac{1}{2!} \lim_{\varepsilon\to 0} \frac{\Gamma(-1+\varepsilon)}{\Gamma(-3+r\varepsilon)} = 3r$$

since $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\frac{1}{\varepsilon} + O(1)$

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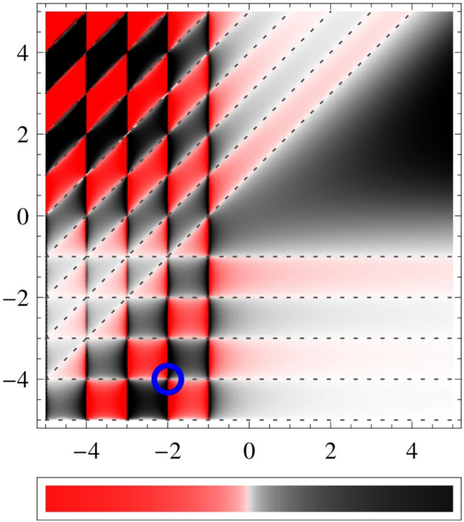

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Sets with a negative number of elements

DEF **Hybrid sets and their subsets**

negative multiplicity

$$\{1, 1, 4 \mid 2, 3, 3\}$$

positive multiplicity

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$$\{\,1,1,4\;|\;2,3,3\,\}$$

positive multiplicity: $1,1,4$ — negative multiplicity: $2,3,3$

$Y \subset X$ if one can repeatedly **remove** elements from X and thus obtain Y or have removed Y.

removing $=$ decreasing the multiplicity of an element with nonzero multiplicity

EG Subsets of $\{1,1,4|2,3,3\}$ include:

(remove 4) $\{4|\}$, $\{1,1|2,3,3\}$

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(remove 4)	$\{4\|\}$,	$\{1, 1\|2, 3, 3\}$
(remove $4, 2, 2$)	$\{2, 2, 4\|\}$,	$\{1, 1\|2, 2, 2, 3, 3\}$

Note that we cannot remove 4 again. $\{4, 4|\}$ is not a subset.

Counting subsets of hybrid sets

- **New sets:** $\underset{\text{all multiplicities } 0,\,1}{\overset{3 \text{ elements:}}{\{1,2,4|\}}}$ or $\underset{\text{all multiplicities } 0,\,-1}{\overset{-4 \text{ elements:}}{\{|1,2,4,5\}}}$

THM Loeb 1992: For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.

EG A usual set like $\{1,2,3|\}$ only has the usual subsets.

Counting subsets of hybrid sets

- **New sets:** 3 elements: $\{1, 2, 4|\}$ (all multiplicities $0, 1$) or -4 elements: $\{|1, 2, 4, 5\}$ (all multiplicities $0, -1$)

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EG ($n = -3$)

- $\left|\binom{-3}{2}\right| = 6$ because the 2-element subsets of $\{|1, 2, 3\}$ are:

$$\{1, 1|\}, \quad \{1, 2|\}, \quad \{1, 3|\}, \quad \{2, 2|\}, \quad \{2, 3|\}, \quad \{3, 3|\}$$

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- $\left|\binom{-3}{-4}\right| = 3$ because the -4-element subsets of $\{|1,2,3\}$ are:

$$\{|1,1,2,3\}, \quad \{|1,2,2,3\}, \quad \{|1,2,3,3\}$$

THM Loeb 1992 — For all integers n and k, $\binom{n}{k} = \{x^k\}(1+x)^n$.

Here, we extract appropriate coefficients:

$$\{x^k\}f(x) := \begin{cases} a_k & \text{if } k \geqslant 0 \\ b_k & \text{if } k < 0 \end{cases}$$

around $x = 0$:

$$f(x) = \sum_{k \geqslant k_0} a_k x^k$$

around $x = \infty$:

$$f(x) = \sum_{k \geqslant k_0} b_{-k} x^{-k}$$

The binomial theorem

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EG

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + O(x^5) \qquad \text{as } x \to 0$$

$$(1+x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O(x^{-6}) \qquad \text{as } x \to \infty$$

Hence, for instance,

$$\binom{-3}{4} = 15, \qquad \binom{-3}{-5} = 6.$$

q-binomial coefficients with integer entries

DEF For all integers n and k,

$$\binom{n}{k}_q := \lim_{a\to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}.$$

$$\binom{-3}{4}_q = \frac{1}{q^{18}}(1-q+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)$$

$$\binom{-3}{-5}_q = \frac{1}{q^7}(1+q^2)(1+q+q^2)$$

S. Formichella, A. Straub
Gaussian binomial coefficients with negative arguments
Annals of Combinatorics, 2019

The q-binomial theorem

THM Formichella S 2019

Suppose $yx = qxy$. For $n, k \in \mathbb{Z}$, $$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

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EG

$$\begin{aligned}(x+y)^{-1} &= y^{-1}(xy^{-1}+1)^{-1} \\ &= y^{-1} \sum_{k \geqslant 0} (-1)^k (xy^{-1})^k \\ &= \sum_{k \geqslant 0} (-1)^k q^{-k(k+1)/2} x^k y^{-k-1}\end{aligned}$$

$\binom{-1}{k}_q$

Counting subsets of hybrid sets, q-version

THM Formichella S 2019

For all $n, k \in \mathbb{Z}$, $$\binom{n}{k}_q = \varepsilon \sum_Y q^{\sigma(Y) - k(k-1)/2}, \quad \varepsilon = \pm 1.$$

The sum is over all k-element subsets Y of the n-element set X_n.

$\varepsilon = 1$ if $0 \leqslant k \leqslant n$. $\varepsilon = (-1)^k$ if $n < 0 \leqslant k$. $\varepsilon = (-1)^{n-k}$ if $k \leqslant n < 0$.

$$X_n := \begin{cases} \{0, 1, \ldots, n-1|\} & \text{if } n \geqslant 0 \\ \{| -1, -2, \ldots, n\} & \text{if } n < 0 \end{cases}$$

$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of y in Y.

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For all $n, k \in \mathbb{Z}$, $\binom{n}{k}_q = \varepsilon \sum_{Y} q^{\sigma(Y) - k(k-1)/2}, \quad \varepsilon = \pm 1.$

The sum is over all k-element subsets Y of the n-element set X_n.

$\varepsilon = 1$ if $0 \leqslant k \leqslant n$. $\varepsilon = (-1)^k$ if $n < 0 \leqslant k$. $\varepsilon = (-1)^{n-k}$ if $k \leqslant n < 0$.

$$X_n := \begin{cases} \{0, 1, \ldots, n-1|\} & \text{if } n \geqslant 0 \\ \{| -1, -2, \ldots, n\} & \text{if } n < 0 \end{cases}$$

$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of y in Y.

EG $n = -3$

The -4-element subsets of $X_{-3} = \{| -1, -2, -3\}$ are:

$\{| -1, -1, -2, -3\}$, $\sigma = 7$; $\{| -1, -2, -2, -3\}$, $\sigma = 8$; $\{| -1, -2, -3, -3\}$, $\sigma = 9$

Hence, $\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1})$. (subtract $\frac{k(k-1)}{2} = 10$)

Conventions for binomial coefficients

Option advertised here:

$$\binom{n}{k} := \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}$$

Alternative:

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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

- used in Mathematica (at least 9+)
- used in Maple (at least 18+)
- used in SageMath (at least 8.0+)

EG

```
Binomial[-3, -5]
> 6
QBinomial[-3, -5, q]
> 0
```

Similarly, `expand(QBinomial(n,k,q))` in Maple 18 results in a division-by-zero error.

Application: Lucas congruences

THM Lucas 1878

Let p be prime. For integers $n, k \geqslant 0$,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where n_i, respectively k_i, are the p-adic digits of n and k.

EG

LHS has 854 digits

$$\binom{3019}{1939} \equiv \binom{10}{1}\binom{7}{12}\binom{10}{6} \equiv 0 \pmod{17}$$

BCB $= 3019$ is prime and happy

Application: Lucas congruences

THM Lucas 1878; Formichella S 2019

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EG

LHS $= 43,758$

$$\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6}\cdots = 3\cdot 5 \equiv 1 \pmod{7}$$

Note the (infinite) 7-adic expansions:

$$-11 = 3 + 5\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + \ldots$$

$$-19 = 2 + 4\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + \ldots$$

Application: q-Lucas congruences


THM Olive 1965, Désarménien 1982

Let $m \geqslant 2$ be an integer. For integers $n, k \geqslant 0$,

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $\begin{aligned} n &= n_0 + n'm \\ k &= k_0 + k'm \end{aligned}$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

B. Adamczewski, J. P. Bell, and E. Delaygue.
Algebraic independence of G-functions and congruences "á la Lucas"
Annales Scientifiques de l'École Normale Supérieure, 2016

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EG

$$\binom{-11}{-19}_q \equiv \binom{3}{2}_q \binom{-2}{-3} = -2(1+q+q^2) \pmod{\Phi_7(q)}$$

- LHS $= \frac{1}{q^{116}}(1+q+2q^2+3q^3+5q^4+\ldots+q^{80})$
- $q=1$ reduces to $\binom{-11}{-19} \equiv -6 \equiv 1 \pmod{7}$.

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Advertisement: More Lucas congruences

Apéry's proof of the irrationality of $\zeta(3)$ centers around:

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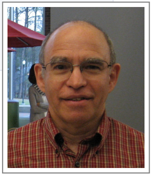

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R. J. McIntosh
A generalization of a congruential property of Lucas.
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

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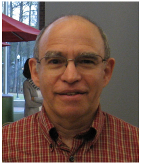


- Gessel's approach generalized by McIntosh (1992)
- $6+6+3$ sporadic Apéry-like sequences are known.

THM Malik–S 2015 Every (known) sporadic sequence satisfies these Lucas congruences modulo every prime.

A. Malik, A. Straub
Divisibility properties of sporadic Apéry-like numbers
Research in Number Theory, Vol. 2, Nr. 1, 2016, p. 1–26

R. J. McIntosh
A generalization of a congruential property of Lucas.
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

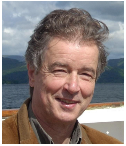
The **Apéry numbers**

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$$

satisfy many interesting properties, including **supercongruences**: $p \geqslant 5$ prime

THM Beukers 1985

$$A(p^r m-1)\equiv A(p^{r-1}m-1) \pmod{p^{3r}}$$



THM Coster 1988

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Application: Apéry number supercongruences

The **Apéry numbers**

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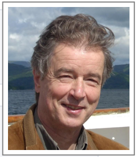

THM Beukers 1985

$$A(p^r m - 1) \equiv A(p^{r-1} m - 1) \pmod{p^{3r}}$$

THM Coster 1988

$$A(p^r m) \equiv A(p^{r-1} m) \pmod{p^{3r}}$$

- Extend $A(n)$ to integers n: $$A(n) = \sum_{k \in \mathbb{Z}} \binom{n}{k}^2 \binom{n+k}{k}^2$$
- It then follows that: $$A(-n) = A(n-1)$$

  Uniform proof (and explanation) of Beukers/Coster supercongruences

Ramanujan-type series for $1/\pi$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \dots$$

The Berndt's have mastered approximating π:

Harvey Berndt: 3.11
Bruce Berndt: 3.13
Sonja Berndt: 3.14

The best kind of π.
(π day 2014, Bruce's office)

Srinivasa Ramanujan
Modular equations and approximations to π
Quarterly Journal of Mathematics, Vol. 45, p. 350–372, 1914

Nayandeep D. Baruah, Bruce C. Berndt, Heng Huat Chan
Ramanujan's series for $1/\pi$: A survey
American Mathematical Monthly, Vol. 116, p. 567–587, 2009

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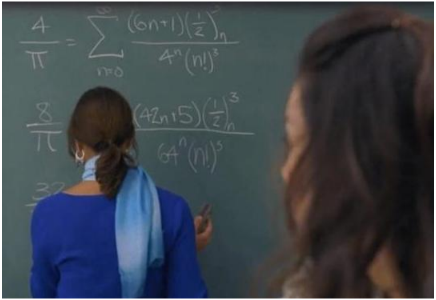
$$= \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(6n+1)\frac{1}{4^n}$$

$$\frac{8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3}(42n+5)\frac{1}{2^{6n}}$$



noticed by Heng Huat Chan with his kids Si Min and Si Ya

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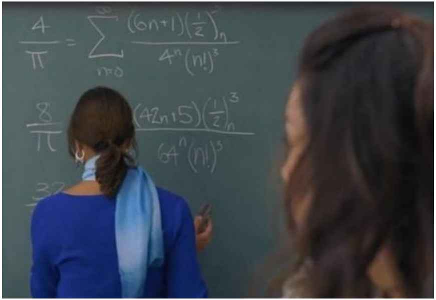

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- First proof of all of Ramanujan's 17 series for $1/\pi$ by Borwein brothers ('87)

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A modular approach to series for $1/\pi$

- Suppose we have a sequence a_n with **modular parametrization**

$$\sum_{n=0}^{\infty} a_n \underbrace{x(\tau)^n}_{\text{modular function}} = \underbrace{f(\tau)}_{\text{modular form}} .$$

- Then:

$$\sum_{n=0}^{\infty} a_n (A + Bn) x(\tau)^n = A f(\tau) + B \frac{x(\tau)}{x'(\tau)} f'(\tau)$$

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (42n+5) \frac{1}{2^{6n}} = \frac{16}{\pi}$$

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FACT For $\tau \in \mathbb{Q}(\sqrt{-d})$:

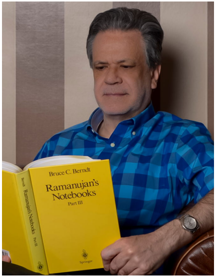
- $x(\tau)$ is an algebraic number.
- $\frac{x(\tau)}{x'(\tau)} f'(\tau) = c f(\tau) + \frac{d}{\pi}$ for algebraic numbers c, d.

p-adic analogs and series for $1/\pi^2$

- Guillera found (and in several cases proved) Ramanujan-type series for $1/\pi^2$ such as

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5(-4)^n}(20n^2+8n+1) = \frac{8}{\pi^2}.$$

For the proven series only WZ style proofs are known.



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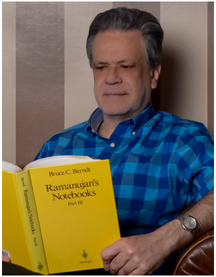

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EG

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3 3^{2n}}(8n+1) = \frac{2\sqrt{3}}{\pi}$$

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3 3^{2n}}(8n+1) \equiv p\left(\frac{-3}{p}\right) \pmod{p^3}$$

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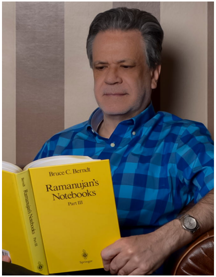

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- Recently: first q-analogs by Guo and Liu (2018) via q-WZ

Krattenthaler: quadratic transformations of q-hypergeometric series

- One of Ramanujan's 17 series for $\frac{1}{\pi}$ and it's p-adic analog:

$$F_N := \sum_{n=0}^{N} \frac{\left(\frac14\right)_n \left(\frac12\right)_n \left(\frac34\right)_n}{n!^3 3^{2n}} (8n+1), \quad F_\infty = \frac{2\sqrt{3}}{\pi}, \quad F_{p-1} \equiv p \left(\frac{-3}{p}\right) \pmod{p^3}$$

THM Guo Zudilin 2018

$$F_N(q) := \sum_{n=0}^{N} q^{2n^2} \frac{(q;q^2)_n^2}{(q^6;q^6)_n^2} \frac{(q;q^2)_{2n}}{(q^2;q^2)_{2n}} [8n+1]_q, \quad F_\infty(q) = \frac{(q^3;q^2)_\infty (q^3;q^6)_\infty}{(q^2;q^2)_\infty (q^6;q^6)_\infty},$$

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Here, n needs to be coprime to 6.

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EG

$$\lim_{q\to 1} \frac{(q^a;q^b)_n}{(q^c;q^d)_n} = \lim_{q\to 1} \prod_{k=0}^{n-1} \frac{1-q^{a+bk}}{1-q^{c+dk}} = \prod_{k=0}^{n-1} \frac{a+bk}{c+dk}$$

so that

$$\lim_{q\to 1} \frac{(q;q^2)_n^2}{(q^6;q^6)_n^2} \frac{(q;q^2)_{2n}}{(q^2;q^2)_{2n}} = \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n}{n!^3 3^{2n}}.$$

Creative microscoping

THM Guo Zudilin 2018

For n coprime to 6:

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The **congruence** follows if one can show the **identity**

$$F_{p-1}(a;q) = q^{-(n-1)/2} [n]_q \left(\frac{-3}{n}\right)$$

in the three cases:

- $q = \zeta$, where $\zeta \neq 1$ is an nth root of unity (hence, congruence mod $[n]_q$)
- $a = q^{-n}$ and $a = q^n$ (hence, congruences mod $1 - aq^n$ and $a - q^n$)

$[n]_q$, $1 - aq^n$, $a - q^n$ are coprime!

□

Creative microscoping

THM Guo Zudilin 2018

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in the three cases:

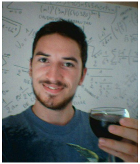
- $q=\zeta$, where $\zeta \neq 1$ is an nth root of unity **(hence, congruence mod $[n]_q$)**
- $a=q^{-n}$ and $a=q^n$ **(hence, congruences mod $1-aq^n$ and $a-q^n$)**

In the latter cases, $F_\infty(q^n;q) = F_{p-1}(q^n;q)$ terminates and the identity follows directly from the series. □

Higher order conjectures

CONJ Gourévitch 2002

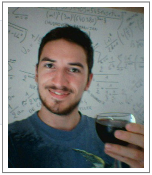
$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{n!^7 2^{6n}} (168n^3 + 76n^2 + 14n + 1) = \frac{32}{\pi^3}$$



- Correspondingly, Zudilin (2009) conjectures that, for $p \geqslant 3$,

$$\sum_{n=0}^{p-1} \frac{\left(\frac{1}{2}\right)_n^7}{n!^7 2^{6n}} (168n^3 + 76n^2 + 14n + 1) \equiv \left(\frac{-1}{p}\right) p^3 \pmod{p^7}.$$

Higher order conjectures



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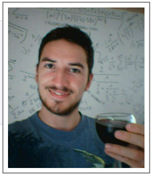
Are there q-analogs? With an additional parameter to permit **creative microscoping**?

Hence a Guo–Zudilin style proof!

Higher order conjectures



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CONJ Cullen 2010

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7 \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!^9 2^{12n}} (43680n^4 + 20632n^3 + 4340n^2 + 466n + 21) = \frac{2^{11}}{\pi^4}$$

- Additional instances of such sequences for $1/\pi^m$ for $m \geqslant 3$?

Happy birthday$_{+86}$, Bruce!

BCB$+1$ day (March 2014)

Samsun (August 2014)

Thank you for being an amazing mentor, supporter, friend and role model!