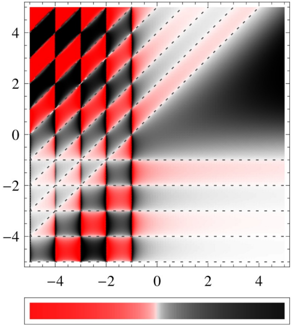
Negative thinking and polynomial analogs

OPSFA-15, Hagenberg, Austria

Armin Straub

July 26, 2019

University of South Alabama



includes joint work with:

Sam Formichella
(University of South Alabama)

Basic q-analogs

IDEA A q-analog reduces to the classical object in the limit $q \to 1$.

DEF

- q-number: $$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots q^{n-1}$$

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- q-factorial: $$[n]_q! = [n]_q [n-1]_q \cdots [1]_q = \frac{(q;q)_n}{(1-q)^n}$$

For q-series fans:

D1

- q-binomial: $$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

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EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

$$= (1-q+q^2) \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

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$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

$\Phi_6(1) = 1$
becomes invisible

Cyclotomic polynomials

DEF The nth cyclotomic polynomial:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k < n \\ (k,n)=1}} (q - \zeta^k) \qquad \text{where } \zeta = e^{2\pi i/n}$$

irreducible polynomial (nontrivial; Gauss!) with **integer** coefficients

- $[n]_q = \dfrac{q^n - 1}{q - 1} = \displaystyle\prod_{\substack{1 < d \leqslant n \\ d \mid n}} \Phi_d(q)$ For primes: $[p]_q = \Phi_p(q)$

EG

$$\begin{aligned}
\Phi_5(q) &= q^4 + q^3 + q^2 + q + 1 \\
\Phi_{21}(q) &= q^{12} - q^{11} + q^9 - q^8 + q^6 - q^4 + q^3 - q + 1 \\
\Phi_{105}(q) &= q^{48} + q^{47} + q^{46} - q^{43} - q^{42} - 2q^{41} - q^{40} - q^{39} \\
&\quad + q^{36} + q^{35} + q^{34} + q^{33} + q^{32} + q^{31} - q^{28} - q^{26} - q^{24} \\
&\quad - q^{22} - q^{20} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} - q^9 \\
&\quad - q^8 - 2q^7 - q^6 - q^5 + q^2 + q + 1
\end{aligned}$$

q-binomials: factored and expanded

LEM factored

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{d=2}^{n} \Phi_d(q)^{\overbrace{\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}} \quad \in \{0,1\}$$

proof

$$[n]_q! = \prod_{m=1}^{n} \prod_{\substack{d|m \\ d>1}} \Phi_d(q) = \prod_{d=2}^{n} \Phi_d(q)^{\lfloor n/d \rfloor}$$

□

- In particular, the q-binomial is a polynomial. (of degree $k(n-k)$)

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EG expanded

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\begin{aligned}\binom{9}{3}_q = q^{18} &+ q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} \\ &+ 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 \\ &+ 4q^4 + 3q^3 + 2q^2 + q + 1\end{aligned}$$

- The coefficients are positive and **unimodal**. Sylvester, 1878

q-binomials: combinatorial

D2

THM

$$\binom{n}{k}_q = \sum_Y q^{w(Y)} \quad \text{where } w(Y) = \overbrace{\sum_j y_j - j}^{\text{"normalized sum of } Y\text{"}}$$

The sum is over all k-element subsets Y of $\{1, 2, \ldots, n\}$.

EG

$$\underbrace{\{1,2\}}_{\to 0}, \quad \underbrace{\{1,3\}}_{\to 1}, \quad \underbrace{\{1,4\}}_{\to 2}, \quad \underbrace{\{2,3\}}_{\to 2}, \quad \underbrace{\{2,4\}}_{\to 3}, \quad \underbrace{\{3,4\}}_{\to 4}$$

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- k-element subsets of n whose normalized sum is m,

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- k-element subsets of n whose normalized sum is m,
- partitions λ of m whose Ferrer's diagram fits in a $k \times (n-k)$ box.

THM The q-binomial satisfies the q-**Pascal rule**:

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$$

D3

q-binomials: three characterizations

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THM Suppose $yx = qxy$ (and that q commutes with x, y). Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

D5

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$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

EG

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from formally inverting D_q

$$\int_0^x f(x) \, \mathrm{d}_q x := (1-q) \sum_{n=0}^{\infty} q^n x f(q^n x)$$

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- The **q-gamma function**:

D6

$$\Gamma_q(s) = \int_0^\infty x^{s-1} e_{1/q}^{-qx} \, \mathrm{d}_q x$$

- $\Gamma_q(s+1) = [s]_q \, \Gamma_q(s)$
- $\Gamma_q(n+1) = [n]_q!$

Can similarly define **q-beta** via a q-Euler integral.

Summary: the q-binomial coefficient

The q-binomial coefficient has a variety of natural characterizations:

- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$
- Via a q-version of **Pascal's rule**
- **Combinatorially**, as the generating function of the element sums of k-subsets of an n-set
- **Geometrically**, as the number of k-dimensional subspaces of $\mathbb{F}_q^n$
- **Algebraically**, via a binomial theorem for noncommuting variables
- **Analytically**, via q-integral representations
- Not touched here: **quantum groups** arising in representation theory and physics

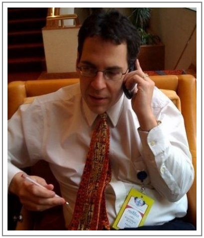
Binomial coefficients with integer entries

$$\binom{-3}{5} = -21, \qquad \binom{-3}{-5} = 6$$

$$\binom{-3.001}{-5.001} \approx 6.004$$

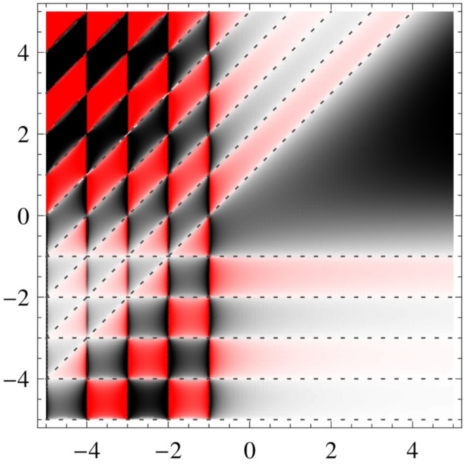
$$\binom{-3.003}{-5.005} \approx 10.03$$

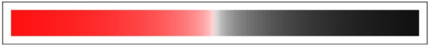
Daniel E. Loeb
Sets with a negative number of elements
Advances in Mathematics, Vol. 91, p.64–74, 1992



1989: Ph.D. at MIT (Rota)
1996+: in mathematical finance

A function in two variables





This scale is also visible along the line $y = 1$.

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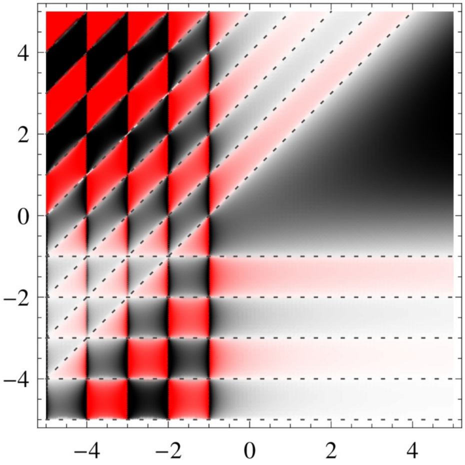

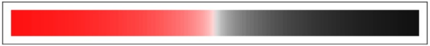

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This is a plot of:

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}$$

Defined and smooth on $\mathbb{R}\backslash\{x = -1, -2, \ldots\}$.

“ … no evidence that the graph of C has ever been plotted before … ”
David Fowler, *American Mathematical Monthly, Jan 1996*

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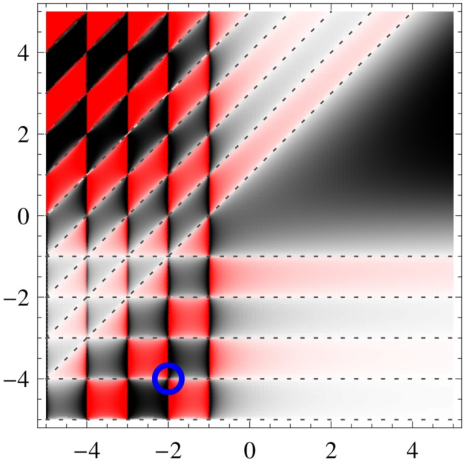

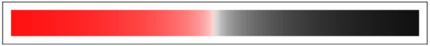

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Directional limits exist at integer points:

$$\lim_{\varepsilon\to 0}\binom{-2+\varepsilon}{-4+r\varepsilon} = \frac{1}{2!}\lim_{\varepsilon\to 0}\frac{\Gamma(-1+\varepsilon)}{\Gamma(-3+r\varepsilon)} = 3r$$

since $\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\frac{1}{\varepsilon} + O(1)$

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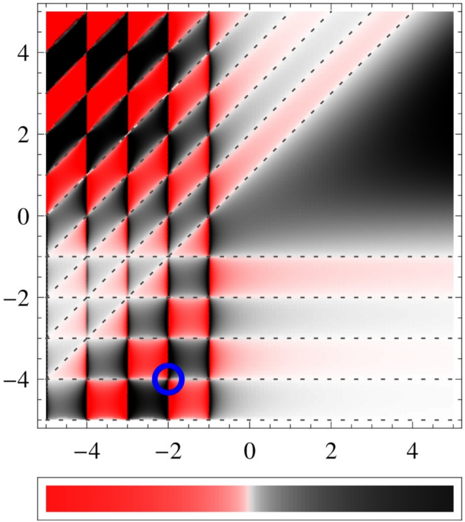

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DEF **Hybrid sets and their subsets**

negative multiplicity

$$\{\, 1, 1, 4 \mid 2, 3, 3 \,\}$$

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$\boldsymbol{Y} \subset \boldsymbol{X}$ if one can repeatedly **remove** elements from X and thus obtain Y or have removed Y.

removing $=$ decreasing the multiplicity of an element with nonzero multiplicity

EG Subsets of $\{1, 1, 4|2, 3, 3\}$ include:

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(remove 4)	$\{4|\}$,	$\{1, 1|2, 3, 3\}$
(remove $4, 2, 2$)	$\{2, 2, 4|\}$,	$\{1, 1|2, 2, 2, 3, 3\}$

Note that we cannot remove 4 again. $\{4, 4|\}$ is not a subset.

Counting subsets of hybrid sets

- **New sets:** $\overset{3 \text{ elements:}}{\underset{\text{all multiplicities } 0,\,1}{\{1,2,4|\}}}$ or $\overset{-4 \text{ elements:}}{\underset{\text{all multiplicities } 0,\,-1}{\{|1,2,4,5\}}}$

THM Loeb 1992: For all integers n and k, the number of k-element subsets of an n-element new set is $\left|\binom{n}{k}\right|$.

EG A usual set like $\{1,2,3|\}$ only has the usual subsets.

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EG $n=-3$

- $\left|\binom{-3}{2}\right| = 6$ because the 2-element subsets of $\{|1,2,3\}$ are:

$$\{1,1|\},\quad \{1,2|\},\quad \{1,3|\},\quad \{2,2|\},\quad \{2,3|\},\quad \{3,3|\}$$

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- $\left|\binom{-3}{-4}\right| = 3$ because the -4-element subsets of $\{|1, 2, 3\}$ are:

$$\{|1, 1, 2, 3\}, \quad \{|1, 2, 2, 3\}, \quad \{|1, 2, 3, 3\}$$

The binomial theorem

THM Loeb 1992

For all integers n and k, $\binom{n}{k} = \{x^k\}(1+x)^n$.

Here, we extract appropriate coefficients:

$$\{x^k\}f(x) := \begin{cases} \boxed{a_k} & \text{if } k \geqslant 0 \\ \boxed{b_k} & \text{if } k < 0 \end{cases}$$

around $x = 0$:

$$f(x) = \sum_{k \geqslant k_0} a_k x^k$$

around $x = \infty$:

$$f(x) = \sum_{k \geqslant k_0} b_{-k} x^{-k}$$

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EG

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 + O(x^5) \qquad \text{as } x \to 0$$

$$(1+x)^{-3} = x^{-3} - 3x^{-4} + 6x^{-5} + O(x^{-6}) \qquad \text{as } x \to \infty$$

Hence, for instance, $$\binom{-3}{4} = 15, \qquad \binom{-3}{-5} = 6.$$

q-binomial coefficients with integer entries

DEF For all integers n and k,

$$\binom{n}{k}_q := \lim_{a \to q} \frac{(a;q)_n}{(a;q)_k (a;q)_{n-k}}.$$

$$\binom{-3}{4}_q = \frac{1}{q^{18}}(1-q+q^2)(1+q+q^2)(1+q+q^2+q^3+q^4)$$

$$\binom{-3}{-5}_q = \frac{1}{q^7}(1+q^2)(1+q+q^2)$$

S. Formichella, A. Straub
Gaussian binomial coefficients with negative arguments
Annals of Combinatorics, 2019

The q-binomial theorem

THM Formichella S 2019

Suppose $yx = qxy$. For $n, k \in \mathbb{Z}$, $$\binom{n}{k}_q = \{x^k y^{n-k}\}(x+y)^n.$$

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around $x = \infty$:
$$f(x) = \sum_{k \geqslant k_0} b_{-k} x^{-k} y^{n+k}$$

EG

$$\begin{aligned}(x+y)^{-1} &= y^{-1}(xy^{-1}+1)^{-1} \\ &= y^{-1} \sum_{k \geqslant 0} (-1)^k (xy^{-1})^k \\ &= \sum_{k \geqslant 0} \boxed{(-1)^k q^{-k(k+1)/2}}\, x^k y^{-k-1}\end{aligned}$$

$$\boxed{\binom{-1}{k}_q}$$

Counting subsets of hybrid sets, q-version

THM Formichella S 2019

For all $n, k \in \mathbb{Z}$, $\binom{n}{k}_q = \varepsilon \sum_Y q^{\boxed{\sigma(Y)} - k(k-1)/2}, \quad \varepsilon = \pm 1.$

The sum is over all k-element subsets Y of the n-element set $\boxed{X_n}$.

$\varepsilon = 1$ if $0 \leqslant k \leqslant n$. $\varepsilon = (-1)^k$ if $n < 0 \leqslant k$. $\varepsilon = (-1)^{n-k}$ if $k \leqslant n < 0$.

$$X_n := \begin{cases} \{0, 1, \ldots, n-1|\} & \text{if } n \geqslant 0 \\ \{| -1, -2, \ldots, n\} & \text{if } n < 0 \end{cases}$$

$$\sigma(Y) := \sum_{y \in Y} M_Y(y) y$$

$M_Y(y)$ is the multiplicity of y in Y.

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EG $n = -3$

The -4-element subsets of $X_{-3} = \{|-1, -2, -3\}$ are:

$$\underset{\sigma = 7}{\{|-1, -1, -2, -3\}}, \quad \underset{\sigma = 8}{\{|-1, -2, -2, -3\}}, \quad \underset{\sigma = 9}{\{|-1, -2, -3, -3\}}$$

Hence, $\binom{-3}{-4}_q = -(q^{-3} + q^{-2} + q^{-1})$. (subtract $\frac{k(k-1)}{2} = 10$)

Conventions for binomial coefficients

Option advertised here:

$$\binom{n}{k} := \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}$$

Alternative:

$$\binom{n}{k} := 0 \qquad \text{if } k < 0$$

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- used in Mathematica (at least 9+)
- used in Maple (at least 18+)

- used in SageMath (at least 8.0+)

EG

```
Binomial[-3, -5]
> 6
QBinomial[-3, -5, q]
> 0
```

Similarly, `expand(QBinomial(n,k,q))` in Maple 18 results in a division-by-zero error.

Application: Lucas congruences

THM Lucas 1878

Let p be prime. For integers $n, k \geqslant 0$,

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots \pmod{p},$$

where n_i, respectively k_i, are the p-adic digits of n and k.

EG

LHS $= 75,582$

$$\binom{19}{11} \equiv \binom{5}{4}\binom{2}{1} = 5 \cdot 2 \equiv 3 \pmod{7}$$

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Formichella S 2019

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EG

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LHS $= 75,582$

EG

$$\binom{-11}{-19} \equiv \binom{3}{2}\binom{5}{4}\binom{6}{6}\binom{6}{6}\cdots = 3 \cdot 5 \equiv 1 \pmod{7}$$

LHS $= 43,758$

Note the (infinite) 7-adic expansions:

$$-11 = 3 + 5 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$
$$-19 = 2 + 4 \cdot 7 + 6 \cdot 7^2 + 6 \cdot 7^3 + \ldots$$

Application: q-Lucas congruences

THM Olive 1965, Désarménien 1982

Let $m \geqslant 2$ be an integer. For integers $n, k \geqslant 0$,

$$\binom{n}{k}_q \equiv \binom{n_0}{k_0}_q \binom{n'}{k'} \pmod{\Phi_m(q)},$$

where $\begin{array}{l} n = n_0 + n'm \\ k = k_0 + k'm \end{array}$ with $n_0, k_0 \in \{0, 1, \ldots, m-1\}$.

B. Adamczewski, J. P. Bell, and E. Delaygue.
Algebraic independence of G-functions and congruences "à la Lucas"
Annales Scientifiques de l'École Normale Supérieure, 2016

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EG

$$\binom{-11}{-19}_q \equiv \binom{3}{2}_q \binom{-2}{-3} = -2(1+q+q^2) \pmod{\Phi_7(q)}$$

- LHS $= \frac{1}{q^{116}}(1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots + q^{80})$
- $q = 1$ reduces to $\binom{-11}{-19} \equiv -6 \equiv 1 \pmod{7}$.

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Advertisement: More Lucas congruences

Apéry's proof of the irrationality of $\zeta(3)$ centers around:

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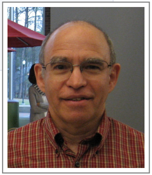
THM Gessel 1982

$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p},$$

where n_i are the p-adic digits of n.



- Gessel's approach generalized by McIntosh (1992)

R. J. McIntosh
A generalization of a congruential property of Lucas.
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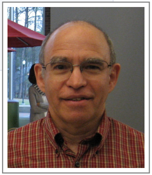


- Gessel's approach generalized by McIntosh (1992)
- $6+6+3$ sporadic Apéry-like sequences are known.

THM Malik–S 2015 Every (known) sporadic sequence satisfies these Lucas congruences modulo every prime.

A. Malik, A. Straub
Divisibility properties of sporadic Apéry-like numbers
Research in Number Theory, Vol. 2, Nr. 1, 2016, p. 1–26

R. J. McIntosh
A generalization of a congruential property of Lucas.
Amer. Math. Monthly, Vol. 99, Nr. 3, 1992, p. 231–238

Application: Apéry number supercongruences

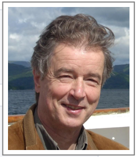
The **Apéry numbers**

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy many interesting properties, including **supercongruences**: $p \geqslant 5$ prime

THM Beukers 1985

$$A(p^r m - 1) \equiv A(p^{r-1} m - 1) \pmod{p^{3r}}$$



THM Coster 1988

$$A(p^r m) \equiv A(p^{r-1} m) \pmod{p^{3r}}$$

The **Apéry numbers**

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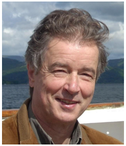

THM Coster 1988

$$A(p^rm)\equiv A(p^{r-1}m)\pmod{p^{3r}}$$

- Extend $A(n)$ to integers n: $$A(n)=\sum_{k\in\mathbb{Z}}\binom{n}{k}^2\binom{n+k}{k}^2$$
- It then follows that: $$A(-n)=A(n-1)$$

  Uniform proof (and explanation) of Beukers/Coster supercongruences

Log-concavity and binomial coefficients

> *So you think nothing new can be said about the binomial coefficients?*
>
> **Victor H. Moll**, $2008 \pm \varepsilon$

Log-concavity of the binomial coefficients

DEF A sequence (a_n) is **log-concave** if $a_n^2 \geqslant a_{n-1}a_{n+1}$.

- Log-concavity (plus positivity) implies unimodality.
- Any concave nonnegative sequence is log-concave.
- Binomial coefficients $\binom{n}{k}$ are log-concave for every fixed n or fixed k.

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DEF Boros Moll

Define the operator $\mathcal{L}$ by $\mathcal{L}(a)_n = a_n^2 - a_{n-1}a_{n+1}$.

(a_n) is **k-log-concave** if $\mathcal{L}^m(a) \geqslant 0$ for $m = 0, 1, \ldots, k$.

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CONJ Boros Moll '04

$\binom{n}{k}$ is ∞-log-concave for every fixed n or fixed k.

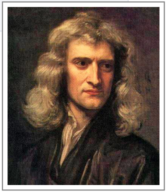
- Proven for fixed n by a theorem of Brändén (2010).
- Still open for fixed k. 5-log-concavity shown by Kauers-Paule (2007).

Real rootedness

(caught the typo! real rottedness)

THM Newton

If the roots of $p(x) = \sum_{k=0}^{n} a_k x^k$ are negative, then (a_k) is log-concave.



In fact: $a_k / \binom{n}{k}$ is log-concave.

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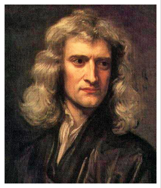

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Conjectured by Fisk, 2008; McNamara-Sagan, 2009; Stanley, 2008:

THM Brändén 2010

If the roots of $p(x) = \sum_{k=0}^{n} a_k x^k$ are negative, then so are the roots of

$$\mathcal{L}[p](x) = \sum_{k=0}^{n} (a_k^2 - a_{k-1}a_{k+1}) x^k.$$

- In particular, then (a_k) is ∞-log-concave.

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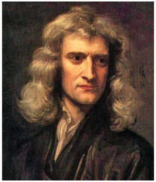

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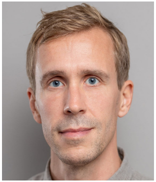

- In particular, then (a_k) is ∞-log-concave.
- Hence, $\binom{n}{k}$ is ∞-log-concave for fixed n

  because $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$.

q-log-concavity

EG $\binom{4}{2}_q = (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$

Coefficients are unimodal, but $\mathcal{L}(1,1,2,1,1) = (1,-1,3,-1,1)$.

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DEF A sequence of polynomials $f_k(q) \in \mathbb{R}[q]$ is **q-log-concave** if

$$\mathcal{L}(f(q)) = f_k(q)^2 - f_{k-1}(q) f_{k+1}(q) \in \mathbb{R}_{\geqslant 0}[q].$$

- q-log-concave implies log-concavity for $q=1$.
- q-log-concave does not imply q-unimodal: $2+5q, \quad 4+4q, \quad 5+2q$.

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THM **Butler 1988** $\binom{n}{k}_q$ is q-log-concave for fixed n.

EG $\binom{n}{k}_q$ is not 2-fold q-log-concave for fixed $n \geqslant 2$. For $n=3$:

$$\begin{array}{rl} & 1\ ,\ q^2+q+1\ ,\ q^2+q+1\ ,\ 1 \\ \mathcal{L}: & 1\ ,\ q^4+2q^3+2q^2+q\ ,\ q^4+2q^3+2q^2+q\ ,\ 1 \\ \mathcal{L}^2: & 1\ ,\ q^8+4q^7+8q^6+10q^5+7q^4+2q^3-q^2-q\ ,\ \ldots \end{array}$$

q-log-concavity, cont'd

CONJ McNamara Sagan 2009 $\binom{n}{k}_q$ is ∞-fold q-log-concave for fixed k.

EG fix k

$$\mathcal{L}\binom{n}{k}_q = \frac{q^{n-k}}{[n]_q}\binom{n}{k}_q\binom{n}{k-1}_q \in \mathbb{R}_{\geqslant 0}[q] \qquad (q\text{-Narayana})$$

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It is not clear that the latter is in $\mathbb{R}_{\geqslant 0}[q]$. (obviously, $\geqslant 0$ when $q=1$)

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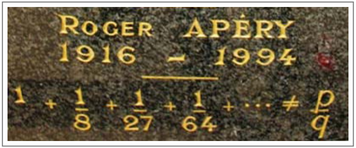
- $\langle n\rangle_q := q^{n-1}+q^{n-3}+\ldots+q^{-(n-1)} = \dfrac{q^n-q^{-n}}{q-q^{-1}}$
- $\left\langle {n \atop k} \right\rangle_q := \dfrac{\langle n\rangle_q!}{\langle k\rangle_q!\,\langle n-k\rangle_q!} = \dfrac{1}{q^{nk-k^2}}\binom{n}{k}_{q^2}$


CONJ McNamara Sagan 2009 $\left\langle {n \atop k} \right\rangle_q$ is ∞-fold q-log-concave for fixed n as well as for fixed k.

Apéry numbers

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational



A. Straub
Supercongruences for polynomial analogs of the Apéry numbers
Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036

A victory for the French peasant. . . *

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2 + 17n + 5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

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proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. $\square$

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> **Alfred van der Poorten** — *A proof that Euler missed. . .* **(1979)**

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Nowadays, there are excellent implementations of this **creative telescoping**, including:

- **HolonomicFunctions** by Koutschan (Mathematica)
- **Sigma** by Schneider (Mathematica)
- **ore_algebra** by Kauers, Jaroschek, Johansson, Mezzarobba (Sage)

(These are just the ones I use on a regular basis. . .)

* Someone's "sour comment" after Henri Cohen's report on Apéry's proof at the '78 ICM in Helsinki.

Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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Q Beukers, Zagier: Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$$ {}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}\,{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2, $$

    with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry numbers

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\substack{\text{modular form} \\ 1+5q+13q^2+23q^3+O(q^4)}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\substack{\text{modular function} \\ q-12q^2+66q^3+O(q^4)}}$$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

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$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- As a consequence, with $z = \sqrt{1 - 34x + x^2}$,

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17 - x - z}{4\sqrt{2}(1 + x + z)^{3/2}}\, {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ 1, 1\end{matrix}\middle| -\frac{1024x}{(1 - x + z)^4}\right).$$

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k + 1$.

Supercongruences for Apéry numbers

- Chowla, Cowles and Cowles (1980) conjectured that, for $p \geqslant 5$,

$$A(p) \equiv 5 \pmod{p^3}.$$

- Gessel (1982) proved that $A(mp) \equiv A(m) \pmod{p^3}$.

THM Beukers, Coster '85, '88

The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

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EG Simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme (1862) showed that, in fact,

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$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$
$$\boxed{\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}}$$
Ljunggren '52

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers.

Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

(a,b,c)	$A(n)$	
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$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open modulo p^3 Amdeberhan–Tauraso '16
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

Non-super congruences are abundant

$$a(mp^r) \equiv a(mp^{r-1}) \pmod{p^r} \qquad \text{(G)}$$

- **realizable** sequences $a(n)$, i.e., for some map $T : X \to X$,

$$a(n) = \#\{x \in X : T^n x = x\}$$ “points of period n”

Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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- $a(n) = \mathrm{trace}(M^n)$ Jänichen '21, Schur '37; also: Arnold, Zarelua

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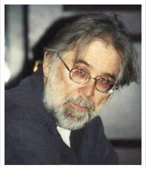
- (G) is equivalent to $\exp\left(\sum_{n=1}^{\infty} \frac{a(n)}{n} T^n\right) \in \mathbb{Z}[[T]]$.

  This is a natural condition in **formal group theory**.



THM Clark 1995

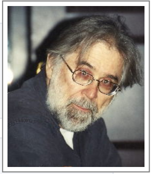
$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} \pmod{\Phi_n(q)^2}$$

proof $a = 2$, $b = 1$

Combinatorially, we have q-Chu-Vandermonde:

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

$\square$



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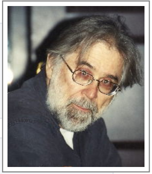
(Note that $\Phi_n(q)$ divides $\binom{n}{k}_q$ unless $k = 0$ or $k = n$.) $\square$

- $\Phi_n(1) = 1$ if n is not a prime power.

q-congruences of Clark and Andrews



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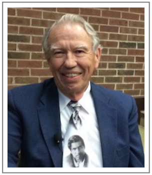
- $\Phi_n(1) = 1$ if n is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$



A q-analog of Ljunggren's congruence

- The following answers Andrews' question to find a q-analog of Wolstenholme's congruence.

THM S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - b(a-b)\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

EG $n=13$, $a=2$, $b=1$

$$\binom{26}{13}_q = \underbrace{1+q^{169}}_{\to 2} - \underbrace{14(q^{13}-1)^2}_{\to 0} + \underbrace{(1+q+\ldots+q^{12})^3}_{\to 13^3} f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\dfrac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- $\binom{ap}{bp} \equiv \binom{a}{b}$ holds modulo p^{3+r} where r is the p-adic valuation of

$$a\,\boxed{b(a-b)\binom{a}{b}}.$$

Jacobsthal 1952

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THM S 2011/18

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THM Zudilin 2019

Extension of above congruence to q-analog of $(p \geqslant 5)$

$$\binom{ap}{bp} \equiv \binom{a}{b} + ab(a-b)p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^4}.$$

Q Creative microscoping à la Guo and Zudilin?

Extra parameter c and congruences modulo, say, $\Phi_n(q)(1-cq^n)(c-q^n)$.

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

EG The first few values are:

$$\begin{aligned}
A(0) &= 1 & A_q(0) &= 1 \\
A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\
A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\
& & &\quad + 9q^6 + 3q^7 + q^8 \\
A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\
& & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18}
\end{aligned}$$

THM
S
2014/18

The q-analog of the Apéry numbers, defined as

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2,$$

satisfies, for any $m \geqslant 0$, $A_q(1) = 1 + 3q + q^2, \quad A(1) = 5$

$$A_q(mn) \equiv A_{q^{m^2}}(n) - \frac{m^2-1}{12}(q^m-1)^2 n^2 A_1(n) \pmod{\Phi_m(q)^3}.$$

THM
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2014/18

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- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.

Q

q-analog and congruences for Almkvist–Zudilin numbers?

$$\sum_{k} (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

(classical supercongruences still open)

Multivariate supercongruences

Q q-analog and congruences for Almkvist–Zudilin numbers?

$$Z(n) = \sum_k (-3)^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$$

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EG (S 2014) The Almkvist–Zudilin numbers are the diagonal Taylor coefficients of

$$\frac{1}{1-(x_1+x_2+x_3+x_4)+27x_1x_2x_3x_4} = \sum_{\boldsymbol{n}\in\mathbb{Z}^4_{\geqslant 0}} Z(\boldsymbol{n})\boldsymbol{x}^{\boldsymbol{n}}$$

CONJ (S 2014) For $p \geqslant 5$, we have the multivariate supercongruences

$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

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$$Z(\boldsymbol{n}p^r) \equiv Z(\boldsymbol{n}p^{r-1}) \pmod{p^{3r}}.$$

CONJ (Gillis, Reznick, Zeilberger 1983) Let $d \geqslant 4$. The following has nonnegative coefficients iff $c \leqslant d!$.

$$\frac{1}{1-(x_1+x_2+\ldots+x_d)+cx_1x_2\cdots x_d}$$

cf. Veronika Pillwein's talk!

- Baryshnikov–Melczer–Pemantle–S (2018): asymptotic positivity for $c < (d-1)^{d-1}$

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

S. Formichella, A. Straub
Gaussian binomial coefficients with negative arguments
Annals of Combinatorics, 2019

A. Straub
A q-analog of Ljunggren's binomial congruence
DMTCS Proceedings: FPSAC 2011, p. 897-902

A. Straub
Supercongruences for polynomial analogs of the Apéry numbers
Proceedings of the American Mathematical Society, Vol. 147, 2019, p. 1023-1036