Interpolated sequences and critical L-values of modular forms

Southern Regional Number Theory Conference:
Modular Curves, Modular Forms, and Hypergeometric Functions

LSU

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$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \qquad f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} \alpha_n q^n$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}$$

$$A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

Joint work with:

Robert Osburn
(University College Dublin)

I

Assorted background

The Riemann zeta function

- The **Riemann zeta function** is the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

- Its zeros and values are fundamental, yet mysterious to this day.

CONJ RH If $\zeta(s) = 0$ then $s \in \{-2, -4, \ldots\}$ or $\operatorname{Re}(s) = \frac{1}{2}$.

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THM Euler 1734: $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots, \qquad \zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}$

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- Open: $\zeta(5)$ is irrational
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- Open: $\zeta(3)/\pi^3$ is irrational
- Open: Catalan's constant $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is irrational

THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi d)^{6h}} \left[\prod_{k=1}^{d} \Gamma\left(\frac{k}{d}\right)^{\left(\frac{-d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $-d < 0$. $\tau_j = \frac{-b_j + \sqrt{-d}}{2a_j}$

throughout, $-d$ is a fundamental discriminant; w is number of roots of unity in $\mathbb{Q}(\sqrt{-d})$

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EG $\mathbb{Q}(\sqrt{-15})$ has discriminant $-d = -15$ and class number $h = 2$.

$$Q_1 = [1, 1, 4] \qquad Q_2 = [2, 1, 2]$$
$$\tau_1 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-15}, \qquad \tau_2 = \tfrac{1}{2}\tau_1$$

$$\frac{1}{\sqrt{2}} |\eta(\tau_1)\eta(\tau_2)|^2 = \frac{1}{30\pi} \left(\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{\Gamma(\frac{7}{15})\Gamma(\frac{11}{15})\Gamma(\frac{13}{15})\Gamma(\frac{14}{15})} \right)^{1/2}$$
$$= \frac{1}{120\pi^3} \Gamma(\tfrac{1}{15})\Gamma(\tfrac{2}{15})\Gamma(\tfrac{4}{15})\Gamma(\tfrac{8}{15})$$

Values of modular forms and functions at CM points

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LEM If $\sigma_1, \sigma_2 \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d})$, then $\frac{\eta(\sigma_1)}{\eta(\sigma_2)}$ is algebraic.

Proof.

- $\sigma_2 = M \cdot \sigma_1$ and $\sigma_1 = N \cdot \sigma_1$ for some $M, N \in \mathrm{GL}_2(\mathbb{Z})$. $[M \neq \mathrm{id}]$
- $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ and $f(N \cdot \tau)$ are modular functions.

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- There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.

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- There is an algebraic relation $\Phi(f(\tau), f(N \cdot \tau)) = 0$.
- Then: $\Phi(f(\sigma_1), f(\sigma_1)) = \Phi(\frac{\eta(\sigma_1)}{\eta(\sigma_2)}, \frac{\eta(\sigma_1)}{\eta(\sigma_2)}) = 0$ □

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LEM If $\sigma_1, \sigma_2 \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d})$, then $\frac{\eta(\sigma_1)}{\eta(\sigma_2)}$ is algebraic.

THM For each $\mathbb{Q}(\sqrt{-d})$, let $\omega_d = \frac{1}{\pi^{1/2}} \left[\prod_{k=1}^{d} \Gamma\left(\frac{k}{d}\right)^{\left(\frac{-d}{k}\right)} \right]^{w/(4h)}$.

For any weight k modular form $f(\tau)$ and any $\sigma \in \mathcal{H} \cap \mathbb{Q}(\sqrt{-d})$, we have $f(\sigma) \in \omega_d^k \bar{\mathbb{Q}}$.

[assuming the functions have algebraic Fourier coefficients]

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EG

$$\eta(i) = \frac{1}{2\pi^{3/4}} \Gamma(\tfrac{1}{4})$$

$$\theta_3(i) = \frac{1}{\sqrt{2}\pi^{3/4}} \Gamma(\tfrac{1}{4})$$

$$\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} = \frac{\eta(\tau)^5}{\eta(\tau/2)^2 \eta(2\tau)^2}$$

$$\theta_3(1 + i\sqrt{2})^4 = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{8\sqrt{2}\pi^3}$$

$$\theta_3\left(-\tfrac{1 - i\sqrt{3}}{2}\right)^4 = \frac{\left(3 - i\sqrt{3}\right)\Gamma^6(\frac{1}{3})}{2^{11/3}\pi^4}.$$

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Question:
CAS implementation?
(We have efficient symbolic-numerical algorithms for values of specific modular functions like j and λ.)

II

Apéry-like sequences

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

  $$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

  satisfy

  $$(n+1)^3 A(n+1) = (2n+1)(17n^2 + 17n + 5)A(n) - n^3 A(n-1).$$

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THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

Proof. The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

Creative telescoping

Goal: a recurrence for $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 =: \sum_{k=0}^{n} A(n,k)$

Let S_n be such that $S_n f(n,k) = f(n+1,k)$.

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

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- Suppose we have $P(n, S_n) \in \mathbb{Q}[n, S_n]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that

$$P(n, S_n) A(n,k) = (S_k - 1) R(n,k) A(n,k).$$

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EG

$$P(n,S_n) = (n+2)^3 S_n^2 - (2n+3)(17n^2+51n+39)S_n + (n+1)^3$$

$$R(n,k) = \frac{4k^4(2n+3)(4n^2-2k^2+12n+3k+8)}{(n-k+1)^2(n-k+2)^2}$$

Automatically obtained using Koutschan's excellent **HolonomicFunctions** package for Mathematica.

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Zagier's search and Apéry-like numbers

- Recurrence for Apéry numbers is the case $(a, b, c) = (17, 5, 1)$ of

$$(n+1)^3 u_{n+1} = (2n+1)(an^2 + an + b)u_n - cn^3 u_{n-1}.$$

Q **Beukers, Zagier** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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Q (Beukers, Zagier): Are there other tuples (a,b,c) for which the solution defined by $u_{-1}=0$, $u_0=1$ is integral?

- Essentially, only 14 tuples (a,b,c) found. (Almkvist–Zudilin)
  - 4 hypergeometric and 4 Legendrian solutions (with generating functions

$${}_3F_2\left(\begin{matrix}\frac{1}{2},\alpha,1-\alpha\\1,1\end{matrix}\middle|4C_\alpha z\right), \qquad \frac{1}{1-C_\alpha z}{}_2F_1\left(\begin{matrix}\alpha,1-\alpha\\1\end{matrix}\middle|\frac{-C_\alpha z}{1-C_\alpha z}\right)^2,$$

  with $\alpha=\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6}$ and $C_\alpha = 2^4, 3^3, 2^6, 2^4\cdot 3^3$)
  - 6 sporadic solutions

- Similar (and intertwined) story for:
  - $(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1}$ (Beukers, Zagier)
  - $(n+1)^3 u_{n+1} = (2n+1)(an^2+an+b)u_n - n(cn^2+d)u_{n-1}$ (Cooper)

The six sporadic Apéry-like numbers

(a,b,c)	$A(n)$	
$(17,5,1)$	$\sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	Apéry numbers
$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Domb numbers
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	Almkvist–Zudilin numbers
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	

Modularity of Apéry-like numbers

- The Apéry numbers $1, 5, 73, 1145, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n \geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}.$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\qquad$ $q - 12q^2 + 66q^3 + O(q^4)$

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FACT Not at all evidently, such a **modular parametrization** exists for all known Apéry-like numbers!

- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation of order $k+1$.

Supercongruences for Apéry numbers

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THM Beukers, Coster '85, '88 — The Apéry numbers satisfy the **supercongruence** $(p \geqslant 5)$

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}.$$

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EG For primes p, simple combinatorics proves the congruence

$$\binom{2p}{p} = \sum_k \binom{p}{k}\binom{p}{p-k} \equiv 1 + 1 \pmod{p^2}.$$

For $p \geqslant 5$, Wolstenholme's congruence shows that, in fact,

$$\binom{2p}{p} \equiv 2 \pmod{p^3}.$$

Supercongruences for Apéry-like numbers

Robert Osburn
(University of Dublin)

Brundaban Sahu
(NISER, India)

- Conjecturally, supercongruences like

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{3r}}$$

hold for all Apéry-like numbers. Osburn–Sahu '09

- Current state of affairs for the six sporadic sequences from earlier:

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$(12,4,16)$	$\sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	Osburn–Sahu–S '16
$(10,4,64)$	$\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$	Osburn–Sahu '11
$(7,3,81)$	$\sum_k (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{n} \frac{(3k)!}{k!^3}$	open; modulo p^3 Amdeberhan–Tauraso '16
$(11,5,125)$	$\sum_k (-1)^k \binom{n}{k}^3 \binom{4n-5k}{3n}$	Osburn–Sahu–S '16
$(9,3,-27)$	$\sum_{k,l} \binom{n}{k}^2 \binom{n}{l} \binom{k}{l} \binom{k+l}{n}$	Gorodetsky '18

III

Interpolations and critical L-values

The Golyshev–Zagier observation

THM
Ahlgren–Ono '00

For any odd prime p, the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a_f(p) \pmod{p^2},$$

with $f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n \geqslant 1} a_f(n) q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

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THM Ahlgren–Ono '00

For any odd prime p, the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv a_f(p) \pmod{p^2},$$

with $f(\tau) = \eta(2\tau)^4\eta(4\tau)^4 = \sum_{n \geqslant 1} a_f(n) q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

THM Zagier '16

$$A(-\tfrac{1}{2}) = \frac{16}{\pi^2} L(f, 2)$$

- Here, $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$ is absolutely convergent for $x \in \mathbb{C}$.
- Predicted by Golyshev based on motivic considerations, the connection of the Apéry numbers with the double covering of a family of K3 surfaces, and the Tate conjecture.

Zagier's six sporadic sequences

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$$

*	(a,b,c)	$C_*(n)$	
A	$(7,2,-8)$	$\sum_{k=0}^{n} \binom{n}{k}^3$	Franel numbers
B	$(9,3,27)$	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	
C	$(10,3,9)$	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	
D	$(11,3,-1)$	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$	Apéry numbers
E	$(12,4,32)$	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	
F	$(17,6,72)$	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$	

Congruences and L-valued interpolations

- For $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$, let $C_*(n)$ be Zagier's sporadic sequence.

THM Beukers, Stienstra '85; OS '18; Kazalicki '18

For $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$, there exists a newform $f_*(\tau) = \sum_{n \geqslant 1} \gamma_{n,*} q^n$ of weight 3, such that, for all primes $p > 2$,

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

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For $*$ one of $\boldsymbol{A}$-$\boldsymbol{F}$, except $\boldsymbol{E}$, there is $\alpha_* \in \mathbb{Z}$ such that

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For sequence $\boldsymbol{E}$, $$\operatorname*{res}_{x=-1/2} C_{\boldsymbol{E}}(x) = \frac{6}{\pi^2} L(f_{\boldsymbol{E}}, 1).$$

Interpolations of Zagier's six sporadic sequences

*	$C_*(n)$	$C_*(x)$
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	$\sum_{k\geqslant 0} \binom{x}{k}^3$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\sum_{k\geqslant 0} (-1)^k 3^{x-3k} \binom{x}{3k} \frac{(3k)!}{k!^3}$
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	$\operatorname{Re} {}_3F_2 \left(\left. \begin{matrix} -x, -x, 1/2 \\ 1, 1 \end{matrix} \right\| 4 \right)$
D	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$	$\sum_{k\geqslant 0} \binom{x}{k}^2 \binom{x+k}{x}$
E	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	$\sum_{k\geqslant 0} \binom{x}{k} \binom{2k}{k} \binom{2(x-k)}{x-k}$
F	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$	$\sum_{k\geqslant 0} (-1)^k 8^{x-k} \binom{x}{k} C_{\boldsymbol{A}}(k)$

The weight 3 newforms of level N_* and their L-values

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2)$$

*	$f_*(\tau)$	N_*	CM	$L(f_*, 2)$	α_*
A	$\frac{\eta(4\tau)^5\eta(8\tau)^5}{\eta(2\tau)^2\eta(16\tau)^2}$	32	$\mathbb{Q}(\sqrt{-2})$	$\frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{64\sqrt{2}\pi}$	8
B	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	$\frac{\Gamma^4\left(\frac{1}{4}\right)}{64\pi}$	8
C	$\eta(2\tau)^3\eta(6\tau)^3$	12	$\mathbb{Q}(\sqrt{-3})$	$\frac{\Gamma^6\left(\frac{1}{3}\right)}{2^{17/3}\pi^2}$	12
D	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	$\frac{\Gamma^4\left(\frac{1}{4}\right)}{64\pi}$	16
E	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$	8	$\mathbb{Q}(\sqrt{-2})$	$\frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{192\pi}$	6
F	$q - 2q^2 + 3q^3 + \ldots$	24	$\mathbb{Q}(\sqrt{-6})$	$\frac{\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right)}{96\sqrt{6}\pi}$	6

L-values of newforms with complex multiplication

THM Ribet '76 — A newform has CM by a quadratic field K (necessarily imaginary and unique) if and only if it comes from a Hecke character of K.

- L-values can then be approached using the work of Damerell ('70)
- L-values for $\boldsymbol{A}$-$\boldsymbol{E}$ computed by Rogers, Wan and Zucker ('15) using binary theta series
- More recent work on explicitly evaluating L-values of CM modular forms by Li, Long, Tu ('18)

W.-C. W. Li, L. Long and F.-T. Tu
Computing special L-values of certain modular forms with complex multiplication
SIGMA, **14**(090), 2018, p. 1–32

M. Rogers, J. G. Wan and I. J. Zucker
Moments of elliptic integrals and critical L-values
Ramanujan Journal, **37**, 2015, p. 113–130

Congruences and interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.

More shortly!

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More shortly!

- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} =: \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

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THM McCarthy, OS '18: Let $N \geqslant 5$ be odd and $k = N - 2$. Then, for all primes $p \geqslant 5$,

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THM OS '18 Let $N \geqslant 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}\left(-\tfrac{1}{2}\right) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are explicit rational numbers, defined recursively.

IV

Selected details and proof sketches

Proof of an interpolation: a simple case

LEM

$$C_{\boldsymbol{D}}(-\tfrac{1}{2}) = \frac{16}{\pi^2} L(f_{\boldsymbol{D}}, 2)$$

Proof. Here, $C_{\boldsymbol{D}}(x) = \sum_{k \geqslant 0} \binom{x}{k}^2 \binom{x+k}{x}$

□

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Proof. Here, $C_{\boldsymbol{D}}(x) = \sum_{k \geqslant 0} \binom{x}{k}^2 \binom{x+k}{x} = {}_3F_2\left(\begin{matrix} -x, -x, x+1 \\ 1, 1 \end{matrix} \middle| 1\right)$. □

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Using hypergeometric identities,

$$C_{\boldsymbol{D}}(-\tfrac{1}{2}) = {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 1\right) = \frac{\Gamma^4(\frac{1}{4})}{4\pi^3}.$$

It remains to show $L(f_{\boldsymbol{D}}, 2) = \dfrac{\Gamma^4(\frac{1}{4})}{64\pi}$.

[Rogers, Wan and Zucker ('15)] □

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- For more challenging cases, a modular parametrization is crucial:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4\lambda(\tau)(1-\lambda(\tau))\right) = {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \middle| \lambda(\tau)\right)^2 = \theta_3(\tau)^4$$

- For $\tau = i$, we get $\lambda(i) = \frac{1}{2}$ and $\theta_3(i)^2 = \frac{\Gamma^2(1/4)}{2\pi^{3/2}}$.

Proof another interpolation: tricky case

LEM Zudilin '18

$$C_{\boldsymbol{F}}(-\tfrac{1}{2}) = \frac{6}{\pi^2} L(f_{\boldsymbol{F}}, 2)$$

- Here, $C_{\boldsymbol{F}}(-\frac{1}{2}) = \dfrac{1}{\sqrt{8}} \displaystyle\sum_{k=0}^{\infty} 2^{-5k} \binom{2k}{k} C_{\boldsymbol{A}}(k)$

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$$C_{\boldsymbol{F}}(-\tfrac{1}{2}) = \frac{6}{\pi^2} L(f_{\boldsymbol{F}}, 2) = \frac{\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{16\sqrt{6}\pi^3}$$

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$$g(z) = \sum_{k=0}^{\infty} z^k \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

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has the modular parametrization [Chan, Tanigawa, Yang, Zudilin ('11)]

$$g\left(\frac{x(\tau)}{(1-x(\tau))^2}\right) = \frac{1}{6}(6E_2(6\tau) + 3E_2(3\tau) - 2E_2(2\tau) - E_2(\tau)),$$

with

$$x(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

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- Setting $\tau = i/\sqrt{6}$ then leads to the result.
(On the other hand, $L(f_{\boldsymbol{F}}, s)$ is a Hecke L-series on the field $\mathbb{Q}(\sqrt{-6})$.)

V

Brown's cellular integrals

Beukers' proof of the irrationality of $\zeta(3)$

$$I_n = (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dxdy$$

$$J_n = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}}\, dxdydw$$

- Beukers showed that

$$I_n = a(n)\zeta(2) + \tilde{a}(n), \qquad J_n = b(n)\zeta(3) + \tilde{b}(n)$$

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- Brown realizes these as period integrals, for $N = 5, 6$, on the moduli space $\mathcal{M}_{0,N}$ of curves of genus 0 with N marked points.

Brown's cellular integrals

THM Brown 2009: Period integrals on $\mathcal{M}_{0,N}$ are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs). (conjectured by Goncharov–Manin, 2004)

- Examples of such integrals can be written as: $(a_i, b_j, c_{ij} \in \mathbb{Z})$

$$\int_{0<t_1<\ldots<t_{N-3}<1} \prod t_i^{a_i}(1-t_j)^{b_j}(t_i-t_j)^{c_{ij}} dt_1 \ldots dt_{N-3}$$

- Typically involve MZVs of all weights $\leqslant N-3$.

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- Typically involve MZVs of all weights $\leqslant N-3$.
- Brown constructs families of integrals $I_\sigma(n)$, for which MZVs of submaximal weight vanish.

  Here, σ are certain ("convergent") permutations in S_N.

N	5	6	7	8	9	10	11
# of σ	1	1	5	17	105	771	7028

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

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EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

One of Brown's cellular integrals

- One of the 17 permutations for $N = 8$ is $\sigma = (8, 3, 6, 1, 4, 7, 2, 5)$.
- Cellular integral $I_\sigma(n) = \int_\Delta f_\sigma^n \, \omega_\sigma$ where $\Delta : 0 < t_2 < \ldots < t_6 < 1$

$$f_\sigma = \frac{(-t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_5)(t_5 - t_6)(t_6 - 1)}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}, \quad \omega_\sigma = \frac{dt_2 dt_3 dt_4 dt_5 dt_6}{(t_3 - t_6)(t_6)(-t_4)(t_4 - 1)(1 - t_2)(t_2 - t_5)}.$$

EG
Panzer: HyperInt

$$I_\sigma(0) = 16\zeta(5) - 8\zeta(3)\zeta(2)$$
$$I_\sigma(1) = 33 I_\sigma(0) - 432\zeta(3) + 316\zeta(2) - 26$$
$$I_\sigma(2) = 8929 I_\sigma(0) - 117500\zeta(3) + \frac{515189}{6}\zeta(2) - \frac{331063}{48}$$

- OGF of $I_\sigma(n)$ satisfies a Picard–Fuchs DE of order 7 (Lairez).
  With 2-dimensional space of analytic solutions at 0.

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$$1, 33, 8929, 4124193, 2435948001, 1657775448033, \ldots$$

One of Brown's cellular integrals, cont'd

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LEM McCarthy, Osburn, S 2018

$$A_\sigma(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}$$

One of Brown's cellular integrals, cont'd

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CONJ McCarthy, Osburn, S 2018 For each $N\geqslant 5$ and convergent σ_N, the leading coefficients $A_{\sigma_N}(n)$ satisfy $(p\geqslant 5)$

$$A_{\sigma_N}(mp^r)\equiv A_{\sigma_N}(mp^{r-1})\pmod{p^{3r}}.$$

For $N=5,6$ these are the supercongruences proved by Beukers and Coster.

One of Brown's cellular integrals, cont'd

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LEM McCarthy, Osburn, S 2018

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THM McCarthy, Osburn, S 2018

For any odd prime p,

$$A_\sigma\left(\frac{p-1}{2}\right) \equiv \gamma(p) \pmod{p^2}.$$

where $\eta^{12}(2\tau) = \sum_{n \geqslant 1} \gamma(n) q^n$ is the unique newform in $S_6(\Gamma_0(4))$.

The Ahlgren–Ono supercongruences

THM Ahlgren–Ono '00

For any odd prime p, the Apéry numbers for $\zeta(3)$ satisfy

$$A\left(\frac{p-1}{2}\right) \equiv \alpha(p) \pmod{p^2},$$

with $\eta(2\tau)^4\eta(4\tau)^4 = \sum_{n\geqslant 1} \alpha(n) q^n$ the unique newform in $S_4(\Gamma_0(8))$.

THM Ahlgren '01

For any prime $p \geqslant 5$, the Apéry numbers for $\zeta(2)$ satisfy

$$B\left(\frac{p-1}{2}\right) \equiv \beta(p) \pmod{p^2},$$

with $\eta(4\tau)^6 = \sum_{n\geqslant 1} \beta(n) q^n$ the unique newform in $S_3(\Gamma_0(16), (\frac{-4}{\cdot}))$.

- conjectured (and proved modulo p) by Beukers '87

An infinite family of supercongruences

- $A_{\sigma_N}(n) = B(n)^{(N-3)/2}$ is one of Brown's sequences for a certain σ_N.
  Here, $B(n)$ are the Apéry numbers for $\zeta(2)$.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m) \in \mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} = \sum_{n \geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, Osburn, S 2018

Let $N \geqslant 5$ be odd. For any prime $p \geqslant 5$,

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THM McCarthy, Osburn, S 2018

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Q Supercongruences for all of Brown's sequences?
Maybe arising from L-series attached to Galois representations?

Hypergeometric supercongruences

THM Kilbourn 2006

$(p \geqslant 3)$

$$ {}_4F_3\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1\end{matrix}\middle| 1\right)_{p-1} \equiv \alpha(p) \pmod{p^3}, $$

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- This result proved the first of 14 related supercongruences conjectured by Rodriguez-Villegas (2001) between
  - truncated hypergeometric series ${}_4F_3$ and
  - Fourier coefficients of modular forms of weight 4.
- 11 of these remained open until recently proved by Long, Tu, Yui, Zudilin (2017).

McCarthy (2010), Fuselier–McCarthy (2016) prove one each; McCarthy (2010) proves "half" of each of the 14.

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Q Can the supercongruences for Brown’s sequences be similarly embedded in the hypergeometric setting?

THM
Osburn, S, Zudilin 2018

$$ {}_6F_5\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \middle| 1 \right)_{p-1} \equiv \lambda(p) \pmod{p^3}, $$

for primes $p > 2$. Here, $\lambda(n)$ are the Fourier coefficients of

$$ \eta(\tau)^8 \eta(4\tau)^4 + 8\eta(4\tau)^{12} = \sum_{n \geqslant 1} \lambda(n) q^n \in S_6(\Gamma_0(8)). $$

- Conjectured by Mortenson based on numerical evidence, which further suggests it holds modulo p^5.

A supercongruence for ${}_6F_5$

THM Osburn, S, Zudilin 2018

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THM Osburn, S, Zudilin 2018

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Q Why do these supercongruences hold?

Promising explanation suggested by Roberts, Rodriguez-Villegas (2017) in terms of gaps between Hodge numbers of an associated motive.

VI

L-value interpolations for cellular integrals

- For an explicit family σ_N of convergent configurations, $A_{\sigma_N}(n) = C_{\boldsymbol{D}}(n)^{(N-3)/2}$.
- For odd $k \geqslant 3$, consider the weight k binary theta series

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2} =: \sum_{n\geqslant 1} \gamma_k(n) q^n.$$

THM McCarthy, OS '18 — Let $N \geqslant 5$ be odd and $k = N - 2$. Then, for all primes $p \geqslant 5$,

$$A_{\sigma_N}(\tfrac{p-1}{2}) \equiv \gamma_k(p) \pmod{p^2}.$$

THM OS '18 — Let $N \geqslant 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}(-\tfrac{1}{2}) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are explicit rational numbers, defined recursively.

THM
OS '18

Let $N \geqslant 5$ be odd and $k = N - 2$. Then,

$$A_{\sigma_N}(-\tfrac{1}{2}) = \frac{\alpha_k}{\pi^{k-1}} L(f_k, k-1),$$

where α_k are given as follows:

$$\alpha_k = 2^{(k+1)/2}(k-2) \begin{cases} 2/r_{(k-1)/2}, & \text{if } k \equiv 1 \pmod{4}, \\ 1/s_{(k-1)/2}, & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

Here, r_n is defined by $r_2 = 1/5$, $r_3 = 0$ and

$$(2n+1)(n-3)r_n = 3 \sum_{k=2}^{n-2} r_k r_{n-k}$$

for $n \geqslant 4$, and s_n is defined by $s_1 = 1/4$, $s_2 = 11/80$, $s_3 = 1/32$ and the same recursion for $n \geqslant 4$.

Interpolations for cellular integrals

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- Since $A_{\sigma_N}(-\frac{1}{2}) = \left(\frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}\right)^{N-3}$, it remains to evaluate $L(f_k, k-1)$.

Evaluating $L(f_k, k-1)$

$$f_k(\tau) = \frac{1}{4} \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{m(k-1)/2} (n - im)^{k-1} q^{n^2+m^2}$$

- The L-value of interest is:

$$L(f_k, k-1) = \frac{1}{4} \sum_{(n,m)\neq(0,0)} (-1)^{m(k-1)/2} \frac{(n-im)^{k-1}}{(n^2+m^2)^{k-1}}$$

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- It remains to evaluate $G_\ell(\tau)$ for $\ell \in \{4, 6\}$ and $\tau \in \{i, 2i\}$.

All critical L-values of f_k

EG
Empirically

$$L(f_5, 4) = \frac{2\pi}{5} L(f_5, 3) = \frac{\pi^2}{5} L(f_5, 2) = \frac{\pi^3}{6} L(f_5, 1)$$

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- By work of Eichler, Shimura and Manin such relations must exist with algebraic numbers.
- $L(f_5,4) = \frac{\pi^2}{5} L(f_5,2)$ follows from a result by Fukuhara, Yang ('13).

Q In principle, such evaluations can be rigorously obtained (Rankin's method). Are there implementations?

Conclusions

- Golyshev and Zagier observed that for

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \qquad f(\tau)=\eta(2\tau)^4\eta(4\tau)^4=\sum_{n\geqslant 1}\alpha_n q^n$$

  the known modular congruences have a continuous analog: weight 4

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- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals

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odd weight k

- Proofs are computational and not satisfactorily uniform

Do all of these have the same motivic explanation?
Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

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  the known modular congruences have a continuous analog: weight 4

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}, \qquad A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f, 2)$$

- We proved that the same phenomenon holds for:
  - all six sporadic sequences of Zagier weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals

  odd weight k

- Proofs are computational and not satisfactorily uniform

  Do all of these have the same motivic explanation?
  Can Zagier's motivic approach (relying on Tate conjecture) be worked out explicitly in these cases?

- Further examples exist. What is the natural framework?

  Apéry-like sequences, CM modular forms, hypergeometric series, ...

- How to characterize the analytic interpolations abstractly?

  We used suitable binomial sums. But the interpolations are not unique! (Some grow like $\sin(\pi x)$ as $x \to i\infty$.)

- Polynomial analogs?

Conclusions

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  - all six sporadic sequences of Zagier — weight 3
  - an infinite family of leading coefficients of Brown's cellular integrals — odd weight k

THM McCarthy, Osburn, S 2018

For any odd prime p,

$$A_\sigma(\tfrac{p-1}{2})\equiv\gamma(p) \pmod{p^2}, \qquad \eta^{12}(2\tau)=\sum_{n\geqslant 1}\gamma(n)q^n\in S_6(\Gamma_0(4))$$

- Here, $A_\sigma(n)=\displaystyle\sum_{\substack{k_1,k_2,k_3,k_4=0\\k_1+k_2=k_3+k_4}}^{n}\prod_{i=1}^{4}\binom{n}{k_i}\binom{n+k_i}{k_i}$.

Question:
Zagier-type interpolation?

THANK YOU!

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VII

Polynomial analogs

Basic q-analogs

- The natural number n has the q-analog:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

In the limit $q \to 1$ a q-analog reduces to the classical object.

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In the limit $q \to 1$ a q-analog reduces to the classical object.

- The q-factorial:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

- The q-binomial coefficient:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \binom{n}{n-k}_q$$

A q-binomial coefficient

EG

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 3 \cdot 5$$

$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$

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$$\begin{aligned}\binom{6}{2}_q &= \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q} \\ &= (1-q+q^2)\underbrace{(1+q+q^2)}_{=[3]_q}\underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}\end{aligned}$$

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$$\binom{6}{2}_q = \frac{(1+q+q^2+q^3+q^5)(1+q+q^2+q^3+q^4)}{1+q}$$
$$= \underbrace{(1-q+q^2)}_{=\Phi_6(q)} \underbrace{(1+q+q^2)}_{=[3]_q} \underbrace{(1+q+q^2+q^3+q^4)}_{=[5]_q}$$

- The cyclotomic polynomial $\Phi_6(q)$ becomes 1 for $q=1$ and hence invisible in the classical world

The coefficients of q-binomial coefficients

- Here's some q-binomials in expanded form:

EG

$$\binom{6}{2}_q = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1$$

$$\binom{9}{3}_q = q^{18} + q^{17} + 2q^{16} + 3q^{15} + 4q^{14} + 5q^{13} + 7q^{12} + 7q^{11} + 8q^{10} + 8q^9 + 8q^8 + 7q^7 + 7q^6 + 5q^5 + 4q^4 + 3q^3 + 2q^2 + q + 1$$

- The degree of the q-binomial is $k(n-k)$.
- All coefficients are positive!
- In fact, the coefficients are unimodal. Sylvester, 1878

A few faces of the q-binomial coefficient

The q-binomial coefficient $\binom{n}{k}_q$

- satisfies a q-version of Pascal's rule, $\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q$,

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A q-analog of Babbage's congruence

- Combinatorially, we again obtain: "q-Chu-Vandermonde"

$$\binom{2n}{n}_q = \sum_{k=0}^{n} \binom{n}{k}_q \binom{n}{n-k}_q q^{(n-k)^2}$$

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THM Clark 1995

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- Note that $\Phi_n(1) = 1$ if n is not a prime power.
- Similar results by Andrews (1999); e.g.:

$$\binom{ap}{bp}_q \equiv q^{(a-b)b\binom{p}{2}} \binom{a}{b}_{q^p} \pmod{[p]_q^2}$$

A q-analog of Ljunggren's congruence

- The following answers the question of Andrews to find a q-analog of Wolstenholme's congruence.

THM S 2011/18

$$\binom{an}{bn}_q \equiv \binom{a}{b}_{q^{n^2}} - (a-b)b\binom{a}{b}\frac{n^2-1}{24}(q^n-1)^2 \pmod{\Phi_n(q)^3}$$

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EG $n = 13$, $a = 2$, $b = 1$

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1+q+\ldots+q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \ldots + q^{132} \in \mathbb{Z}[q]$.

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- Note that $\frac{n^2-1}{24}$ is an integer if $(n,6)=1$.
- Ljunggren's classical congruence holds modulo p^{3+r} with r the p-adic valuation of Jacobsthal '52

$$(a-b)ab\binom{a}{b}.$$

A q-version of the Apéry numbers

- A symmetric q-analog of the Apéry numbers:

$$A_q(n) = \sum_{k=0}^{n} q^{(n-k)^2} \binom{n}{k}_q^2 \binom{n+k}{k}_q^2$$

This is an explicit form of a q-analog of Krattenthaler, Rivoal and Zudilin (2006).

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- The first few values are:

$$\begin{aligned} A(0) &= 1 & A_q(0) &= 1 \\ A(1) &= 5 & A_q(1) &= 1 + 3q + q^2 \\ A(2) &= 73 & A_q(2) &= 1 + 3q + 9q^2 + 14q^3 + 19q^4 + 14q^5 \\ & & &\quad + 9q^6 + 3q^7 + q^8 \\ A(3) &= 1445 & A_q(3) &= 1 + 3q + 9q^2 + 22q^3 + 43q^4 + 76q^5 \\ & & &\quad + 117q^6 + \ldots + 3q^{17} + q^{18} \end{aligned}$$

THM
S
2014/18

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THM S 2014/18

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- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.

THM S 2014/18

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- Gorodetsky (2018) recently proved q-congruences implying the stronger congruences $A(p^r n) \equiv A(p^{r-1} n)$ modulo p^{3r}.
- q-analog and congruences for Almkvist–Zudilin numbers? (classical supercongruences still open)

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