Interpolated sequences and critical L-values of modular forms

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$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \qquad f(\tau)=\eta(2\tau)^4\eta(4\tau)^4=\sum_{n\geqslant 1}\alpha_n q^n$$

$$1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$$

$$A(\tfrac{p-1}{2}) \equiv \alpha_p \pmod{p^2}$$
$$A(-\tfrac{1}{2}) = \tfrac{16}{\pi^2} L(f,2)$$

Joint work with:

Robert Osburn
(University College Dublin)

A victory for the French peasant...*

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

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proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

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Nowadays, there are excellent implementations of this **creative telescoping**, including:

- **HolonomicFunctions** by Koutschan (Mathematica)
- **Sigma** by Schneider (Mathematica)
- **ore_algebra** by Kauers, Jaroschek, Johansson, Mezzarobba (Sage)

(These are just the ones I use on a regular basis...)

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Zagier's search and Apéry-like numbers

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$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}, \qquad (a, b, c) = (11, 3, -1).$$

Q **Beukers** Are there other tuples (a, b, c) for which the solution defined by $u_{-1} = 0$, $u_0 = 1$ is integral?

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- Apart from degenerate cases, Zagier found 6 sporadic integer solutions:

*	$C_*(n)$	*	$C_*(n)$
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	**D**	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	**E**	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	**F**	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$

THM Ahlgren–Ono 2000

For primes $p > 2$, the Apéry numbers for $\zeta(3)$ satisfy

$$A(\tfrac{p-1}{2}) \equiv a_f(p) \pmod{p^2},$$

with $f(\tau) = \eta(2\tau)^4 \eta(4\tau)^4 = \sum_{n \geqslant 1} a_f(n) q^n \in S_4(\Gamma_0(8))$.

conjectured (and proved modulo p) by Beukers '87

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THM Zagier 2016

$$A(-\tfrac{1}{2}) = \frac{16}{\pi^2} L(f,2)$$

- Here, $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$ is absolutely convergent for $x \in \mathbb{C}$.
- Predicted by Golyshev based on motivic considerations, the connection of the Apéry numbers with the double covering of a family of K3 surfaces, and the Tate conjecture.

D. Zagier
Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017

L-value interpolations, cont'd

- Zagier found 6 sporadic integer solutions $C_*(n)$ to: $\quad *$ one of $\boldsymbol{A}$-$\boldsymbol{F}$

$$(n+1)^2 u_{n+1} = (an^2+an+b)u_n - cn^2 u_{n-1} \qquad u_{-1}=0, u_0=1$$

THM 1985 - 2019

There exists a weight 3 newform $f_*(\tau) = \sum_{n\geqslant 1} \gamma_{n,*} q^n$, so that

$$C_*\left(\tfrac{p-1}{2}\right) \equiv \gamma_{p,*} \pmod{p}.$$

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- $\boldsymbol{C}$, $\boldsymbol{D}$ proved by Beukers–Stienstra ('85); $\boldsymbol{A}$ follows from their work
- $\boldsymbol{E}$ proved using a result Verrill ('10); $\boldsymbol{B}$ through p-adic analysis
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For sequence $\boldsymbol{E}$, $\quad \operatorname*{res}_{x=-1/2} C_{\boldsymbol{E}}(x) = \dfrac{6}{\pi^2} L(f_{\boldsymbol{E}}, 1).$

$$C_*(-\tfrac{1}{2}) = \frac{\alpha_*}{\pi^2} L(f_*, 2)$$

*	$C_*(n)$	$f_*(\tau)$	N_*	CM	α_*
A	$\sum_{k=0}^{n} \binom{n}{k}^3$	$\frac{\eta(4\tau)^5\eta(8\tau)^5}{\eta(2\tau)^2\eta(16\tau)^2}$	32	$\mathbb{Q}(\sqrt{-2})$	8
B	$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	8
C	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$	$\eta(2\tau)^3\eta(6\tau)^3$	12	$\mathbb{Q}(\sqrt{-3})$	12
D	$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$	$\eta(4\tau)^6$	16	$\mathbb{Q}(\sqrt{-1})$	16
E	$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$	$\eta(\tau)^2\eta(2\tau)\eta(4\tau)\eta(8\tau)^2$	8	$\mathbb{Q}(\sqrt{-2})$	6
F	$\sum_{k=0}^{n} (-1)^k 8^{n-k} \binom{n}{k} C_{\boldsymbol{A}}(k)$	$q - 2q^2 + 3q^3 + \ldots$	24	$\mathbb{Q}(\sqrt{-6})$	6

Challenge: $A \equiv B$

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EG Osburn-S 2018

For primes $p > 2$ and $n = \frac{p-1}{2}$,

$$\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3} \equiv \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \pmod{p}.$$

$C_{\boldsymbol{B}}(n)$ $C_{\boldsymbol{D}}(n)$

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- Our proof of this congruence relies on finding (?!) the **identity**

$$\text{RHS} = \sum_{k=0}^{n} (-1)^k \binom{3n+1}{n-k} \binom{n+k}{k}^3.$$

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LEM Osburn-S-Zudilin 2018

For primes $p > 2$ and $n = \frac{p-1}{2}$,

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k}^3 \binom{n}{k}^3 \left(1 - 3k(2H_k - H_{n+k} - H_{n-k})\right) \equiv \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 \pmod{p^2}.$$

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$$\text{RHS} = \frac{(-1)^n}{2} \sum_{k=0}^{n} \binom{n+k}{n} \binom{2n-k}{n} \binom{n}{k}^4 \times \left(2 + (n-2k)(5H_k - 5H_{n-k} - H_{n+k} + H_{2n-k})\right).$$

Creative telescoping

Goal: a recurrence for $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 =: \sum_{k=0}^{n} A(n,k)$

Let S_n be such that $S_n f(n,k) = f(n+1,k)$.

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A. K. Peters, Ltd., 1st edition, 1996

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- Suppose we have $P(n, S_n) \in \mathbb{Q}[n, S_n]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that
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- Then: $$P(n, S_n) \sum_{k \in \mathbb{Z}} A(n,k) = 0$$

EG

$$P(n, S_n) = (n+2)^3 S_n^2 - (2n+3)(17n^2+51n+39) S_n + (n+1)^3$$
$$R(n,k) = \frac{4k^4(2n+3)(4n^2-2k^2+12n+3k+8)}{(n-k+1)^2(n-k+2)^2}$$

Automatically obtained using Koutschan's excellent **HolonomicFunctions** package for Mathematica.

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$$P(x, S_x)A(x) = \frac{8}{\pi^2}(2x+3)\sin^2(\pi x)$$

for all complex x, where $P(x, S_x)$ is Apéry's recurrence operator.

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$$\begin{aligned} P(x, S_x) \sum_{k=0}^{K-1} A(x, k) &= R(x, K) A(x, K) - R(x, 0) A(x, 0) \\ &= R(x, K) A(x, K) \end{aligned}$$

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- For the $\zeta(2)$ Apéry numbers $B(n)$, we use $B(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}$.

  However:

  - The series diverges if $\mathrm{Re}\, x < -1$.
  - $Q(x, S_x)B(x) = 0$ where $Q(x, S_x)$ is Apéry's recurrence operator.

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diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG ($\boldsymbol{C}$)

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EG **(C)**

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| 4\right)$$

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We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix}-x, -x, \frac{1}{2}\\ 1, 1\end{matrix}\right| 4\right)$.

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EG ($\boldsymbol{E}$)

$$C_{\boldsymbol{E}}(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

Challenge: Interpolating sequences

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- For Apéry numbers $A(n)$, Zagier used $A(x) = \sum_{k=0}^{\infty} \binom{x}{k}^2 \binom{x+k}{k}^2$.

diverges for $n \notin \mathbb{Z}_{\geqslant 0}$

EG (C)

$$C_{\boldsymbol{C}}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} = {}_3F_2\left(\left.\begin{matrix} -n, -n, \frac{1}{2} \\ 1, 1 \end{matrix}\right| 4\right)$$

We use the interpolation $C_{\boldsymbol{C}}(x) = \operatorname{Re}\, {}_3F_2\left(\left.\begin{matrix} -x, -x, \frac{1}{2} \\ 1, 1 \end{matrix}\right| 4\right)$.

EG (E)

$$C_{\boldsymbol{E}}(n) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} = \binom{2n}{n} {}_3F_2\left(\left.\begin{matrix} -n, -n, \frac{1}{2} \\ \frac{1}{2}-n, 1 \end{matrix}\right| -1\right)$$

This has a simple pole at $n = -\frac{1}{2}$.

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EG

$$C(n) = \sum_{\substack{k_1,k_2,k_3,k_4=0 \\ k_1+k_2=k_3+k_4}}^{n} \prod_{i=1}^{4} \binom{n}{k_i}\binom{n+k_i}{k_i}.$$

How to compute $C(-\frac{1}{2})$?

- RE: order 4, degree 15
- DE: order 7, degree 17
  (2 analytic solutions)

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How to compute $C(-\frac{1}{2})$?

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  (2 analytic solutions)

THM McCarthy, Osburn, S 2018 For any odd prime p,

$$C(\tfrac{p-1}{2}) \equiv \gamma(p) \pmod{p^2}, \qquad \eta^{12}(2\tau) = \sum_{n \geqslant 1} \gamma(n) q^n \in S_6(\Gamma_0(4))$$

Q Is there a Zagier-type interpolation?

Challenge: computing values of $\eta(\tau)$ at CM points

Q How to efficiently compute $\eta(\tau)$ for quadratic irrationalities τ?

Lots of papers would benefit from a CAS implementation!

- **Dedekind eta function**: the prototypical modular form of weight $\frac{1}{2}$

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n \geqslant 1} (1 - e^{2\pi i n \tau}).$$

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EG

$$\eta(i) = \frac{1}{2\pi^{3/4}} \Gamma\left(\tfrac{1}{4}\right)$$

$$\theta_3(i) = \frac{1}{\sqrt{2}\pi^{3/4}} \Gamma\left(\tfrac{1}{4}\right)$$

$$\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2/2} = \frac{\eta(\tau)^5}{\eta(\tau/2)^2 \eta(2\tau)^2}$$

$$\theta_3(1 + i\sqrt{2})^4 = \frac{\Gamma^2\left(\frac{1}{8}\right)\Gamma^2\left(\frac{3}{8}\right)}{8\sqrt{2}\pi^3}$$

$$\theta_3\left(-\frac{1 - i\sqrt{3}}{2}\right)^4 = \frac{\left(3 - i\sqrt{3}\right)\Gamma^6\left(\frac{1}{3}\right)}{2^{11/3}\pi^4}.$$

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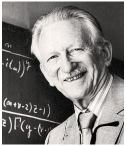
THM Chowla–Selberg 1967

$$\prod_{j=1}^{h} a_j^{-6} |\eta(\tau_j)|^{24} = \frac{1}{(2\pi d)^{6h}} \left[\prod_{k=1}^{d} \Gamma\left(\tfrac{k}{d}\right)^{\left(\frac{-d}{k}\right)} \right]^{3w}$$

where the product is over reduced binary quadratic forms $[a_j, b_j, c_j]$ of discriminant $-d < 0$. $\tau_j = \frac{-b_j + \sqrt{-d}}{2a_j}$

here, $-d$ is a fundamental discriminant; w is number of roots of unity in $\mathbb{Q}(\sqrt{-d})$



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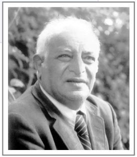

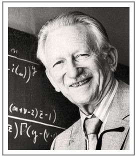

- The $|\eta(\tau_j)|$ only differ by an algebraic factor:
  - $\tau_2 = M \cdot \tau_1$ for some $M \in \mathrm{GL}_2(\mathbb{Z})$.
  - $f(\tau) = \frac{\eta(\tau)}{\eta(M \cdot \tau)}$ is a modular function with $f(\tau_1) = \frac{\eta(\tau_1)}{\eta(\tau_2)}$.

Challenge: computing values of $\eta(\tau)$ at CM points

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FACT f a modular function, τ_0 a quadratic irrationality
$\Longrightarrow \quad f(\tau_0)$ is an algebraic number.

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FACT f a modular function, τ_0 a quadratic irrationality
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- $A \cdot \tau_0 = \tau_0$ for some $A \in \mathrm{GL}_2(\mathbb{Z})$
- Two modular functions are related by a **modular equation**:

$$P(f(A \cdot \tau), f(\tau)) = 0$$

- Hence: $f(\tau_0)$ is a root of $P(x, x) = 0$.

BUT Complexity of modular equations increases very quickly.

Q How to efficiently compute $\eta(\tau)$ for quadratic irrationalities τ?

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FACT f a modular function, τ_0 a quadratic irrationality
$\implies$ $f(\tau_0)$ is an algebraic number.

- $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$ $\quad q = e^{2\pi i\tau}$
- **Modular polynomial** $\Phi_N \in \mathbb{Z}[x, y]$ such that $\Phi_N(j(N\tau), j(\tau)) = 0$.

EG

$$\begin{aligned}\Phi_2(x, y) = {} & x^3 + y^3 - x^2y^2 + 2^4 \cdot 3 \cdot 31(x^2 + xy^2) \\ & - 2^4 \cdot 3^4 \cdot 5^3(x^2 + y^2) + 3^4 \cdot 5^3 \cdot 4027xy \\ & + 2^8 \cdot 3^7 \cdot 5^6(x + y) - 2^{12} \cdot 3^9 \cdot 5^9\end{aligned}$$

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Φ_N is $O(N^3 \log N)$ bits

$$\begin{aligned}\Phi_{11}(x,y) &= x^{12} + y^{12} - x^{11}y^{11} + 8184x^{11}y^{10} - 28278756x^{11}y^9 \\ &\quad + \ldots \text{several pages} \ldots + \\ &\quad + 392423345094527654908696 \ldots \text{100 digits} \ldots 000\end{aligned}$$

$\Phi_{11}(x, y)$ due to Kaltofen–Yui (1984)

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Other options for evaluating $f(\tau_0)$:

- via PSLQ/LLL and rigorous bounds
- via class field theory (Shimura reciprocity)

EG class field theory To evaluate $j(\frac{1+\sqrt{-23}}{2})$, we determine its Galois conjugates:

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EG class field theory

To evaluate $j(\frac{1+\sqrt{-23}}{2})$, we determine its Galois conjugates:

$$\left(x - j(\tfrac{1+\sqrt{-23}}{2})\right)\left(x - j(\tfrac{1+\sqrt{-23}}{4})\right)\left(x - j(\tfrac{-1+\sqrt{-23}}{4})\right)$$
$$= x^3 + 3491750x^2 - 5151296875x + 12771880859375$$

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

D. McCarthy, R. Osburn, A. Straub
Sequences, modular forms and cellular integrals
Mathematical Proceedings of the Cambridge Philosophical Society, 2018

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Interpolated sequences and critical L-values of modular forms
Chapter 14 of the book: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*
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A modular supercongruence for ${}_6F_5$: An Apéry-like story
Annales de l'Institut Fourier, Vol. 68, Nr. 5, 2018, p. 1987-2004

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Arithmetic and topology of differential equations
Proceedings of the 2016 ECM, 2017