Sums of powers of binomials, their Apéry limits, and Franel's suspicions

Mathematics Colloquium
Dalhousie University

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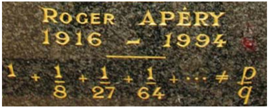
University of South Alabama

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational



based on joint work(s) with:

Marc Chamberland
(Grinnell College)

Wadim Zudilin
(Radboud University)

Mit der Tür ins Haus fallen... Falling into the house with the door...

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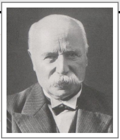

THM Stoll '97 — $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

OPEN Is that recurrence of minimal order?

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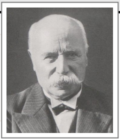

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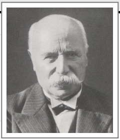

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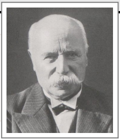
The Apéry limits are:

$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)}\right)^s \in \pi^{2j}\mathbb{Q}_{>0}$$



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Moreover, $A_j^{(s)}(n)$ with $0 \leqslant 2j < s$ are linearly independent, so that any telescoping recurrence has order at least $\lfloor \frac{s+1}{2} \rfloor$.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. $\square$

> *After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence satisfies the recurrence.*
>
> **Alfred van der Poorten** — *A proof that Euler missed...* **(1979)**

Goal

A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

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- Suppose we have $P(n,N) \in \mathbb{Q}[n,N]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that:

  $$P(n,N)a(n,k) = (K-1)R(n,k)a(n,k)$$

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Background: Creative telescoping

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EG

$$P(n,N) = (n+2)^3 N^2 - (2n+3)(17n^2+51n+39)N + (n+1)^3$$

$$R(n,k) = \frac{4k^4(2n+3)(4n^2-2k^2+12n+3k+8)}{(n-k+1)^2(n-k+2)^2}$$

$R(n,k)$ is the **certificate** of the **telescoping recurrence** operator $P(n,N)$.

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Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

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$$\lambda^d + c_{d-1} \lambda^{d-1} + \cdots + c_1 \lambda + c_0 = \prod_{k=1}^{d} (\lambda - \lambda_k)$$

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THM
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Suppose the $|\lambda_k|$ are distinct. Then, for any solution u_n,

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for some $k \in \{1, \ldots, d\}$, unless u_n is eventually zero.

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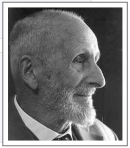

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Suppose, in addition, $p_0(n) \neq 0$ for all $n \geqslant 0$.
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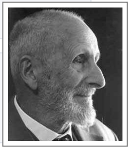

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EG Kooman 1989: For $u_{n+2} - 2u_{n+1} + (1 + \frac{1}{n^2})u_n = 0$, we have $\lambda_1 = \lambda_2 = 1$. However, (P) does not hold for any real u_n.

There are two complex solutions asymptotic to n^r with $r = \exp(\pm \pi i/3)$.

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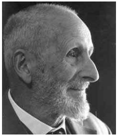

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For $\alpha_n u_{n+2} + (\alpha_{n+1} - \alpha_n) u_{n+1} - \alpha_{n+1} u_n = 0$, we have $\lambda_1, \lambda_2 = \pm 1$.
However, (P) holds for all u_n with RHS $= 1$. $\qquad \alpha_n = 1 + \frac{(-1)^n}{n}$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

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$$C(n) = \gamma A(n) + B(n) \quad \text{with} \quad \lim_{n\to\infty} \frac{C(n+1)}{C(n)} = (1-\sqrt{2})^4.$$

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? Tools to construct the solutions guaranteed by Perron's theorem?

A motivating example

- The (central) **Delannoy numbers** $A(n) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$ satisfy

$$(n+1)u_{n+1} = 3(2n+1)u_n - nu_{n-1} \qquad A(-1) = 0, A(0) = 1$$

count lattice paths from $(0,0)$ to (n,n) using the steps $(0,1)$, $(1,0)$ and $(1,1)$

$$A(n) = 1, 3, 13, 63, 321, 1683, 8989, 48639, \ldots$$

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$$B(n) = 0, 1, \frac{9}{2}, \frac{131}{6}, \frac{445}{4}, \frac{34997}{60}, \frac{62307}{20}, \frac{2359979}{140}, \ldots$$

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EG bonus

For $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{3k}{n}$, determine and prove the Apéry limits.
This is one of many cases conjectured by Almkvist, van Straten and Zudilin (2008) for CY DE's. Can we establish all these limits in a uniform fashion?

Approaches to proving Apéry limits

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1. Via explicit expressions: (Apéry, '78)

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5. Via continued fractions (for recurrences of order 2)

Continued fractions and Apéry limits

$$C = \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \qquad (b_n \neq 0)$$

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proof The n-th convergent is $C_n := \frac{a_1}{b_1+} \frac{a_2}{b_2+} \cdots \frac{a_n}{b_n} = \frac{B(n)}{A(n)}$. □

Continued fractions and Apéry limits

$$C = \frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots := \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} \qquad (b_n \neq 0)$$

THM 1-1 correspondence between CFs and order 2 recurrences, such that the value of the CF is an Apéry limit: $C = \lim_{n\to\infty} \frac{B(n)}{A(n)}$

- Here, $A(n), B(n)$ are the solutions to $u_n = b_n u_{n-1} + a_n u_{n-2}$
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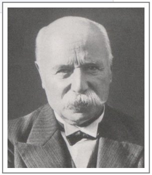
Apéry limit and equivalent CF:

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{1}{(2x+1)-}\,\frac{1^2}{3(2x+1)-}\,\frac{2^2}{5(2x+1)-}\cdots = \frac{1}{2}\ln\left(1 + \frac{1}{x}\right)$$

Franel numbers

DEF Franel 1894 $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



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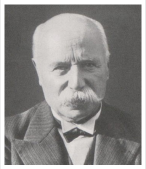

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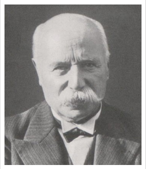

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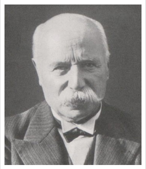

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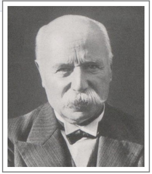

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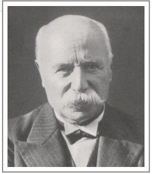

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CONJ Franel, 1895

The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ and degree $s - 1$. (spoiler: the degree part is not true)

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If true, the degree grows like $s^3/24$.

- Verified at least for $s \leqslant 20$.

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How to prove lower bounds for orders of recurrences?

EG

- $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
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  For Apéry numbers: $\mu = (1+\sqrt{2})^4$.
  For Franel numbers: $\mu = 2^s$. Not helpful!

Solutions to the Franel number recurrences

THM
S-Zudilin '21

Any telescoping recurrence for $\sum_{k=0}^{n}\binom{n}{k}^{s}$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

(fine print: for large enough n)

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4. $A^{(s)}(n,t) = \left(\frac{\pi t}{\sin(\pi t)}\right)^s \sum_{k=0}^{n} \binom{n}{k-t}^s$ and so $P(n,N) A^{(s)}(n,t) = O(t^s)$. $\square$

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$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)}\right)^s \in \pi^{2j}\mathbb{Q}_{>0}$$

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$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j} \mathbb{Q}_{>0}$$

- Our proof is based on showing locally uniform convergence in t of

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Apéry limits for Franel numbers

THM S-Zudilin '21: Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

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"poof" For large n and $k \approx n/2$,

$$\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right) \approx \prod_{j=1}^{\infty}\left(1-\frac{t}{j}\right)\left(1+\frac{t}{j}\right) = \frac{\sin(\pi t)}{\pi t}.$$

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EG $j = 1$

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$$B^{(s)}(n) = \frac{A_1^{(s)}(n)}{A_1^{(s)}(1)} = 0, 1, \ldots$$

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EG $j = 2$

$$\lim_{n\to\infty} \frac{C^{(s)}(n)}{A^{(s)}(n)} = \frac{12}{s(s+1)(s+2)(s+3)} \frac{s(5s+2)}{4}\zeta(4)$$

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Telescoping version of Franel's conjecture

THM S-Zudilin '21: Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ has order at least $\lfloor \frac{s+1}{2} \rfloor$.

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Here, and below, we assume that n is large enough.

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  if the minimal-order recurrence is telescoping. True at least for $s \leqslant 30$.

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- Order could be reduced by a different representation such as:

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When does creative telescoping fall short?

EG
Paule, Schorn '95

Consider $S_d(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{dk}{n}$.

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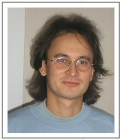

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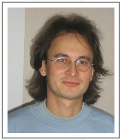

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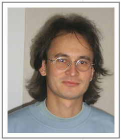

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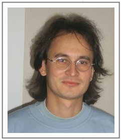

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> "Studying a huge number of practical applications one is tempted to conjecture that Zeilberger's algorithm always returns the recurrence with minimal order."
> **Peter Paule, Markus Schorn**, *Journal of Symbolic Computation, 1995*

- Let $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$ be Zagier's sporadic sequence **C**. $1, 3, 15, 93, \ldots$

THM
Zagier '09

$$\underbrace{\frac{\eta(2\tau)^6\eta(3\tau)}{\eta(\tau)^3\eta(6\tau)^2}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta(\tau)^4\eta(6\tau)^8}{\eta(2\tau)^8\eta(3\tau)^4}\right)^n}_{\text{modular function}}$$

$f(\tau) = 1 + 3q + 3q^2 + 3q^3 + O(q^4)$ $x(\tau) = q - 4q^2 + 10q^3 + O(q^4)$ $q = e^{2\pi i\tau}$

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- Context:

$f(\tau)$	modular form of weight k
$x(\tau)$	modular function
$y(x)$	such that $y(x(\tau)) = f(\tau)$

Then $y(x)$ satisfies a linear differential equation $Ly = 0$ of order $k + 1$.

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- Solutions to $Ly=\operatorname{rat}(x)$ are of the form $y(x)$ times an Eichler integral of

$$h(\tau)=\left(\frac{Dx(\tau)}{x(\tau)}\right)^{k+1}\frac{\operatorname{rat}(x(\tau))}{f(\tau)}$$ (a modular form of weight $k+2$) (Yang '07)

$D=q\frac{\mathrm{d}}{\mathrm{d}q}$

If $\sum_{n\geqslant 1}c_nq^n$ is a modular form of weight $k+2$, then $\sum_{n\geqslant 1}\frac{c_n}{n^{k+1}}q^n$ is an Eichler integral.

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$f(\tau) = 1 + 3q + 3q^2 + 3q^3 + O(q^4)$ $\quad x(\tau) = q - 4q^2 + 10q^3 + O(q^4)$ $\quad q = e^{2\pi i \tau}$

- $F(x) := \sum_{n \geqslant 0} A(n) x^n \quad \Longrightarrow \quad F(x(\tau)) = f(\tau)$
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Modularity and Apéry limits

- Let $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$ be Zagier's sporadic sequence **C**. $1, 3, 15, 93, \ldots$

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characteristic roots $1, 9$

$F(x), G(x)$ have radius of convergence $R = \frac{1}{9}$.
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characteristic roots $1, 9$ $\quad$ $x(\tau) = \frac{1}{9}$ for $\tau = 0$ or $q = 1$

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THM Zagier '09

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Conclusions and some open questions

- Applications of Apéry limits:
  - Irrationality proofs for $\zeta(2)$ and $\zeta(3)$
  - Explicitly construct the solutions guaranteed by Perron's theorem
  - Continued fractions
  - Prove lower bounds on orders of recurrences **new!**

Conclusions and some open questions

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  - Prove lower bounds on orders of recurrences **new!**

- Many open questions! For instance:
  - Cusick '89 and Stoll '97 construct recurrences for Franel numbers. Can these constructions produce telescoping recurrences?
  - What can we learn from other families of binomial sums?

    Also, it would be nice to simplify some of the technical steps in the arguments.
  - Can we (uniformly) establish the conjectural Apéry limits for CY DE's?
  - Can we explain when CT falls short? And algorithmically "fix" this issue?

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

M. Chamberland, A. Straub
Apéry limits: Experiments and proofs
American Mathematical Monthly, Vol. 128, Nr. 9, 2021, p. 811-824

A. Straub, W. Zudilin
Sums of powers of binomials, their Apéry limits, and Franel's suspicions
International Mathematics Research Notices, to appear, 2022. arXiv:2112.09576