Sums of powers of binomials, their Apéry limits, and Franel's suspicions

Special Session on The Intersection of Number Theory and Combinatorics
AMS Fall Central Sectional Meeting, El Paso

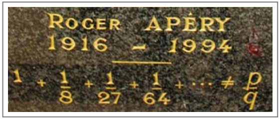
Armin Straub

September 18, 2022

University of South Alabama

CONJ $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent over $\mathbb{Q}$.

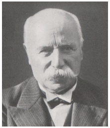
- Apéry (1978): $\zeta(3)$ is irrational
- Open: $\zeta(5)$ is irrational
- Open: $\zeta(3)$ is transcendental
- Open: $\zeta(3)/\pi^3$ is irrational



based on joint work(s) with:

Marc Chamberland
(Grinnell College)

Wadim Zudilin
(Radboud University)



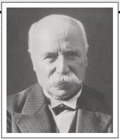
CONJ Franel, 1895 — The minimal recurrence for $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ has order $\lfloor \frac{s+1}{2} \rfloor$.

THM Stoll '97 — $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

OPEN Is that recurrence of minimal order?

Mit der Tür ins Haus fallen...

Falling into the house with the door...

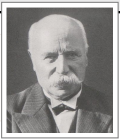


CONJ Franel, 1895 — The minimal recurrence for $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ has order $\lfloor \frac{s+1}{2} \rfloor$.

THM Stoll '97 — $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

OPEN Is that recurrence of minimal order?

THM S-Zudilin '21 — Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by certain sequences $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.



CONJ Franel, 1895

The minimal recurrence for $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ has order $\lfloor \frac{s+1}{2} \rfloor$.

THM Stoll '97

$A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

OPEN Is that recurrence of minimal order?

THM S-Zudilin '21

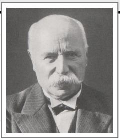
Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by certain sequences $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

The Apéry limits are:

$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)}\right)^s \in \pi^{2j}\mathbb{Q}_{>0}$$

CONJ Franel, 1895

The minimal recurrence for $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ has order $\lfloor \frac{s+1}{2} \rfloor$.



THM Stoll '97

$A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

OPEN Is that recurrence of minimal order?

THM S-Zudilin '21

Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by certain sequences $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

The Apéry limits are:

$$\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j}\mathbb{Q}_{>0}$$

Moreover, $A_j^{(s)}(n)$ with $0 \leqslant 2j < s$ are linearly independent, so that any telescoping recurrence has order at least $\lfloor \frac{s+1}{2} \rfloor$.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

- The **Apéry numbers** $1, 5, 73, 1445, \ldots$

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

THM Apéry '78: $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational.

proof The same recurrence is satisfied by the "near"-integers

$$B(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

Then, $\frac{B(n)}{A(n)} \to \zeta(3)$. But too fast for $\zeta(3)$ to be rational. □

> *After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence satisfies the recurrence.*
>
> **Alfred van der Poorten** — *A proof that Euler missed...* **(1979)**

Background: Creative telescoping

Goal

A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=:\, a(n,k)}$

N, K shift operators in n and k: $N a(n,k) = a(n+1,k)$

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

Goal A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

- Suppose we have $P(n,N) \in \mathbb{Q}[n,N]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that:

$$P(n,N)a(n,k) = (K-1)R(n,k)a(n,k)$$

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

Background: Creative telescoping

Goal A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

- Suppose we have $P(n,N) \in \mathbb{Q}[n,N]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that:

$$P(n,N)a(n,k) = (K-1)R(n,k)a(n,k) \quad = \quad b(n,k+1) - b(n,k)$$

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
A = B
A. K. Peters, Ltd., 1st edition, 1996

Goal

A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

- Suppose we have $P(n,N) \in \mathbb{Q}[n,N]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that:

$$P(n,N)a(n,k) = (K-1)R(n,k)a(n,k) \quad = \quad b(n,k+1) - b(n,k)$$

- Then: $$P(n,N) \sum_{k \in \mathbb{Z}} a(n,k) = 0$$

Assuming $\lim_{k \to \pm\infty} b(n,k) = 0$.

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
$A = B$
A. K. Peters, Ltd., 1st edition, 1996

Goal A **telescoping recurrence** for $\sum_{k=0}^{n} \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a(n,k)}$

N, K shift operators in n and k: $Na(n,k) = a(n+1,k)$

- Suppose we have $P(n,N) \in \mathbb{Q}[n,N]$ and $R(n,k) \in \mathbb{Q}(n,k)$ so that:

$$P(n,N)a(n,k) = (K-1)R(n,k)a(n,k) \quad = \quad b(n,k+1) - b(n,k)$$

- Then: $P(n,N) \sum_{k \in \mathbb{Z}} a(n,k) = 0$ Assuming $\lim_{k \to \pm\infty} b(n,k) = 0$.

EG

$$P(n,N) = (n+2)^3 N^2 - (2n+3)(17n^2+51n+39)N + (n+1)^3$$

$$R(n,k) = \frac{4k^4(2n+3)(4n^2-2k^2+12n+3k+8)}{(n-k+1)^2(n-k+2)^2}$$

$R(n,k)$ is the **certificate** of the **telescoping recurrence** operator $P(n,N)$.

Marko Petkovsek, Herbert S. Wilf and Doron Zeilberger
A = B
A. K. Peters, Ltd., 1st edition, 1996

Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

$$u_{n+d} + p_{d-1}(n)\, u_{n+d-1} + \cdots + p_1(n)\, u_{n+1} + p_0(n)\, u_n = 0$$

Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

$$u_{n+d} + p_{d-1}(n)\, u_{n+d-1} + \cdots + p_1(n)\, u_{n+1} + p_0(n)\, u_n = 0$$

- If $\lim_{n\to\infty} p_k(n) = c_k$, then the **characteristic polynomial** is:

$$\lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + c_0 = \prod_{k=1}^{d}(\lambda - \lambda_k)$$

Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

$$u_{n+d} + p_{d-1}(n)\, u_{n+d-1} + \cdots + p_1(n)\, u_{n+1} + p_0(n)\, u_n = 0$$

- If $\lim_{n\to\infty} p_k(n) = c_k$, then the **characteristic polynomial** is:

$$\lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1 \lambda + c_0 = \prod_{k=1}^{d} (\lambda - \lambda_k)$$

THM
Poincaré 1885

Suppose the $|\lambda_k|$ are distinct. Then, for any solution u_n,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda_k \qquad \text{(P)}$$

for some $k \in \{1, \ldots, d\}$, unless u_n is eventually zero.

Background: Poincaré and Perron

- Normalized general homogeneous linear recurrence of order d:

$$u_{n+d} + p_{d-1}(n)\, u_{n+d-1} + \cdots + p_1(n)\, u_{n+1} + p_0(n)\, u_n = 0$$

- If $\lim_{n\to\infty} p_k(n) = c_k$, then the **characteristic polynomial** is:

$$\lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + c_0 = \prod_{k=1}^{d} (\lambda - \lambda_k)$$

THM Poincaré 1885

Suppose the $|\lambda_k|$ are distinct. Then, for any solution u_n,

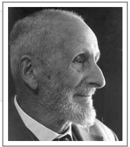
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda_k \qquad \text{(P)}$$

for some $k \in \{1, \ldots, d\}$, unless u_n is eventually zero.

THM Perron 1909

Suppose, in addition, $p_0(n) \neq 0$ for all $n \geqslant 0$.

Then, for each λ_k, there exists a u_n such that (P) holds.



Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1} = 0, u_0 = 1$: Apéry numbers $A(n)$ $\quad 1, 5, 73, 1445, 33001, \ldots$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\qquad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\qquad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\quad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\quad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM (Apéry '78)

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\quad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\quad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM Apéry '78

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

- Characteristic polynomial $n^2 - 34n + 1$ has roots $(1 \pm \sqrt{2})^4 \approx 33.97, 0.0294$.

$A(n)$, $B(n)$ grow like $(1+\sqrt{2})^{4n}$.

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\qquad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\qquad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM Apéry '78

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

- Characteristic polynomial $n^2 - 34n + 1$ has roots $(1 \pm \sqrt{2})^4 \approx 33.97, 0.0294$.

  $A(n)$, $B(n)$ grow like $(1+\sqrt{2})^{4n}$.

- By Perron's theorem, there is a (unique) solution

$$C(n) = \gamma A(n) + B(n) \quad \text{with} \quad \lim_{n\to\infty} \frac{C(n+1)}{C(n)} = (1-\sqrt{2})^4.$$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\quad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\quad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM Apéry '78

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

- Characteristic polynomial $n^2-34n+1$ has roots $(1\pm\sqrt{2})^4 \approx 33.97, 0.0294$.

  $A(n)$, $B(n)$ grow like $(1+\sqrt{2})^{4n}$.

- By Perron's theorem, there is a (unique) solution

$$C(n) = \gamma A(n) + B(n) \quad \text{with} \quad \lim_{n\to\infty} \frac{C(n+1)}{C(n)} = (1-\sqrt{2})^4.$$

$$\downarrow$$

$$0 = \gamma + \lim_{n\to\infty} \frac{B(n)}{A(n)}$$

Another look at Apéry's recurrence and limit

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\quad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\quad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM Apéry '78

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

- Characteristic polynomial $n^2 - 34n + 1$ has roots $(1 \pm \sqrt{2})^4 \approx 33.97, 0.0294$.

  $A(n)$, $B(n)$ grow like $(1+\sqrt{2})^{4n}$.

- By Perron's theorem, there is a (unique) solution

$$C(n) = \gamma A(n) + B(n) \quad \text{with} \quad \lim_{n\to\infty} \frac{C(n+1)}{C(n)} = (1-\sqrt{2})^4.$$

$$\downarrow$$

$$0 = \gamma + \lim_{n\to\infty} \frac{B(n)}{A(n)}$$

COR $A(n)\zeta(3) - 6B(n)$ is "*Perron's small solution*".

This is a small linear form in 1 and $\zeta(3)$.

- Apéry's recurrence has order 2 and degree 3:

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}.$$

- $u_{-1}=0, u_0=1$: Apéry numbers $A(n)$ $\qquad 1, 5, 73, 1445, 33001, \ldots$
- $u_0=0, u_1=1$: 2nd solution $B(n)$ $\qquad 0, 1, \frac{117}{8}, \frac{62531}{216}, \frac{11424695}{1728}, \ldots$

THM Apéry '78

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$$

- Characteristic polynomial $n^2 - 34n + 1$ has roots $(1\pm\sqrt{2})^4 \approx 33.97, 0.0294$.

  $A(n)$, $B(n)$ grow like $(1+\sqrt{2})^{4n}$.

- By Perron's theorem, there is a (unique) solution

$$C(n) = \gamma A(n) + B(n) \quad \text{with} \quad \lim_{n\to\infty} \frac{C(n+1)}{C(n)} = (1-\sqrt{2})^4.$$

$$\downarrow$$

$$0 = \gamma + \lim_{n\to\infty} \frac{B(n)}{A(n)}$$

COR $A(n)\zeta(3) - 6B(n)$ is "*Perron's small solution*".

This is a small linear form in 1 and $\zeta(3)$.

? Tools to construct the solutions guaranteed by Perron's theorem?

Approaches to proving Apéry limits

Q How to prove $\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$?

1. Via explicit expressions: (Apéry, '78)

$$B(n) = \frac{1}{6} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

Q How to prove $\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$?

1. Via explicit expressions: (Apéry, '78)

$$B(n) = \frac{1}{6} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

2. Via integral representations: (Beukers, '79)

$$(-1)^n \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} \mathrm{d}x \mathrm{d}y \mathrm{d}z = A(n)\zeta(3) - 6B(n)$$

Approaches to proving Apéry limits

Q How to prove $\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$?

1. Via explicit expressions: (Apéry, '78)

$$B(n) = \frac{1}{6} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

2. Via integral representations: (Beukers, '79)

$$(-1)^n \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-xy)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = A(n)\zeta(3) - 6B(n)$$

3. Via hypergeometric series representations: (Gutnik, '79)

$$-\frac{1}{2} \sum_{t=1}^{\infty} R_n'(t) = A(n)\zeta(3) - 6B(n), \quad \text{where } R_n(t) = \left(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)} \right)^2$$

Approaches to proving Apéry limits

Q How to prove $\lim_{n\to\infty} \frac{B(n)}{A(n)} = \frac{\zeta(3)}{6}$?

1. Via explicit expressions: (Apéry, '78)

$$B(n) = \frac{1}{6} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{j=1}^{n} \frac{1}{j^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right)$$

2. Via integral representations: (Beukers, '79)

$$(-1)^n \int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n z^n (1-z)^n}{(1-(1-xy)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = A(n)\zeta(3) - 6B(n)$$

3. Via hypergeometric series representations: (Gutnik, '79)

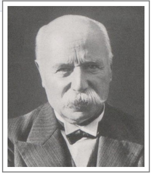
$$-\frac{1}{2} \sum_{t=1}^{\infty} R_n'(t) = A(n)\zeta(3) - 6B(n), \quad \text{where } R_n(t) = \left(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)} \right)^2$$

4. Via modular forms (Beukers '87, Zagier '03, Yang '07)
5. Via continued fractions (for recurrences of order 2)

Franel numbers

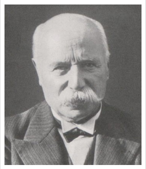
DEF
Franel 1894

$$A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$$ are the (generalized) Franel numbers.



Franel numbers

DEF Franel 1894: $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



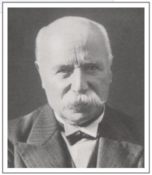
- $A^{(1)}(n) = 2^n$

  $u_{n+1} = 2u_n$

Franel numbers

DEF Franel 1894

$A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



- $A^{(1)}(n) = 2^n$

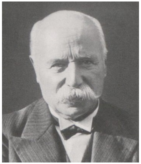
  $u_{n+1} = 2u_n$

- $A^{(2)}(n) = \binom{2n}{n}$

  $(n+1)u_{n+1} = 2(2n+1)u_n$

Franel numbers

DEF Franel 1894 $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



- $A^{(1)}(n) = 2^n$

  $u_{n+1} = 2u_n$

- $A^{(2)}(n) = \binom{2n}{n}$

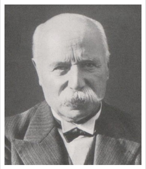
  $(n+1)u_{n+1} = 2(2n+1)u_n$

- $A^{(3)}(n) = 1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, \ldots$

  $(n+1)^2 u_{n+1} = (7n^2 + 7n + 2)u_n + 8n^2 u_{n-1}$ (Franel, 1894)

Franel numbers

DEF Franel 1894 $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



- $A^{(1)}(n) = 2^n$

  $u_{n+1} = 2u_n$

- $A^{(2)}(n) = \binom{2n}{n}$

  $(n+1)u_{n+1} = 2(2n+1)u_n$

- $A^{(3)}(n) = 1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, \ldots$

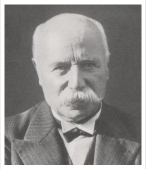
  $(n+1)^2 u_{n+1} = (7n^2 + 7n + 2)u_n + 8n^2 u_{n-1}$ (Franel, 1894)

- $A^{(4)}(n) = 1, 2, 18, 164, 1810, 21252, 263844, 3395016, 44916498, \ldots$

  $(n+1)^3 u_{n+1} = 2(2n+1)(3n^2 + 3n + 1)u_n + 4n(16n^2 - 1)u_{n-1}$ (Franel, 1895)

Franel numbers

DEF Franel 1894: $A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ are the (generalized) Franel numbers.



- $A^{(1)}(n) = 2^n$

  $u_{n+1} = 2u_n$

- $A^{(2)}(n) = \binom{2n}{n}$

  $(n+1)u_{n+1} = 2(2n+1)u_n$

- $A^{(3)}(n) = 1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, \ldots$

  $(n+1)^2 u_{n+1} = (7n^2+7n+2)u_n + 8n^2 u_{n-1}$ (Franel, 1894)

- $A^{(4)}(n) = 1, 2, 18, 164, 1810, 21252, 263844, 3395016, 44916498, \ldots$

  $(n+1)^3 u_{n+1} = 2(2n+1)(3n^2+3n+1)u_n + 4n(16n^2-1)u_{n-1}$ (Franel, 1895)

CONJ Franel, 1895: The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ and degree $s-1$. (spoiler: the degree part is not true)

Franel's conjecture

CONJ Franel, 1895

The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ and degree $s-1$.

Franel's conjecture

CONJ Franel, 1895: The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ and degree $s-1$.

- Perlstadt '86: order 3 recurrences for $s = 5, 6$ of degrees $6, 9$
  computed using MACSYMA and creative telescoping

Franel's conjecture

CONJ Franel, 1895 — The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ ~~and degree $s-1$~~.

- Perlstadt '86: order 3 recurrences for $s = 5, 6$ of degrees $6, 9$

  computed using MACSYMA and creative telescoping

THM Stoll '97 — $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

Cusick '89 also constructs such recurrences.

OPEN Is that recurrence of minimal order?

Franel's conjecture

CONJ Franel, 1895: The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ ~~and degree $s-1$~~.

- Perlstadt '86: order 3 recurrences for $s = 5, 6$ of degrees $6, 9$

  computed using MACSYMA and creative telescoping

THM Stoll '97: $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

Cusick '89 also constructs such recurrences.

OPEN Is that recurrence of minimal order?

CONJ Bostan '21: The minimal recurrence for $A^{(s)}(n)$ has order $m = \lfloor \frac{s+1}{2} \rfloor$ and

$$\text{degree} = \begin{cases} \frac{1}{3}m(m^2-1)+1, & \text{for even } s, \\ \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{2}{3}m + \frac{(-1)^m-1}{4}, & \text{for odd } s. \end{cases}$$

If true, the degree grows like $s^3/24$.

- Verified at least for $s \leqslant 20$.

  using `MinimalRecurrence` from the `LREtools` Maple package

Franel's conjecture

CONJ Franel, 1895: The minimal recurrence for $A^{(s)}(n)$ has order $\lfloor \frac{s+1}{2} \rfloor$ ~~and degree $s-1$~~.

- Perlstadt '86: order 3 recurrences for $s = 5, 6$ of degrees $6, 9$

  computed using MACSYMA and creative telescoping

THM Stoll '97: $A^{(s)}(n)$ satisfies a recurrence of order $\lfloor \frac{s+1}{2} \rfloor$.

Cusick '89 also constructs such recurrences.

OPEN Is that recurrence of minimal order?

CONJ Bostan '21: The minimal recurrence for $A^{(s)}(n)$ has order $m = \lfloor \frac{s+1}{2} \rfloor$ and

$$\text{degree} = \begin{cases} \frac{1}{3}m(m^2-1)+1, & \text{for even } s, \\ \frac{1}{3}m^3 - \frac{1}{2}m^2 + \frac{2}{3}m + \frac{(-1)^m - 1}{4}, & \text{for odd } s. \end{cases}$$

If true, the degree grows like $s^3/24$.

- Verified at least for $s \leqslant 20$.

  using `MinimalRecurrence` from the `LREtools` Maple package

- Goal: The minimal **telescoping** recurrence for $A^{(s)}(n)$ has order $\geqslant \lfloor \frac{s+1}{2} \rfloor$.

How to prove lower bounds for orders of recurrences?

EG

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? . . . and higher degree

How to prove lower bounds for orders of recurrences?

EG

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? . . . and higher degree

- For fixed sequence, order 1 can be ruled out using `Hyper`, (Petkovšek '92)
  an algorithm to compute order 1 (right) factors of recurrence operators.

How to prove lower bounds for orders of recurrences?

EG

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? ... and higher degree

- For fixed sequence, order 1 can be ruled out using `Hyper`, (Petkovšek '92)
  an algorithm to compute order 1 (right) factors of recurrence operators.
- There are algorithms for fixed recurrence operators (Beke 1894, Bronstein '94, Zhou–van Hoeij '19, ...)
  for computing factors of differen(tial/ce) operators.

How to prove lower bounds for orders of recurrences?

EG

- $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? . . . and higher degree

- For fixed sequence, order 1 can be ruled out using `Hyper`, (Petkovšek '92)
  an algorithm to compute order 1 (right) factors of recurrence operators.
- There are algorithms for fixed recurrence operators (Beke 1894, Bronstein '94, Zhou–van Hoeij '19, . . .)
  for computing factors of differen(tial/ce) operators.
- For Franel numbers, order 1 can be ruled out for all $s \geqslant 3$ (Yuan–Lu–Schmidt '08)
  using congruential properties.

How to prove lower bounds for orders of recurrences?

EG

- $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? ...and higher degree

- For fixed sequence, order 1 can be ruled out using `Hyper`, (Petkovšek '92)
  an algorithm to compute order 1 (right) factors of recurrence operators.
- There are algorithms for fixed recurrence operators (Beke 1894, Bronstein '94, Zhou–van Hoeij '19, ...)
  for computing factors of differen(tial/ce) operators.
- For Franel numbers, order 1 can be ruled out for all $s \geqslant 3$ (Yuan–Lu–Schmidt '08)
  using congruential properties.
- If $A(n+1)/A(n) \to \mu$ for $\mu \in \bar{\mathbb{Q}}$ of degree d, then $A(n)$ cannot satisfy a recurrence over $\mathbb{Q}$ of order less than d. (McIntosh '89)

How to prove lower bounds for orders of recurrences?

EG

- $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$: recurrence of order 2 (Apéry '78)
- $\sum_{k=0}^{n} \binom{n}{k}^s$: recurrence of order $\lfloor \frac{s+1}{2} \rfloor$ (Stoll '97)

Could there be recurrences of lower order? ... and higher degree

- For fixed sequence, order 1 can be ruled out using `Hyper`, (Petkovšek '92)
  an algorithm to compute order 1 (right) factors of recurrence operators.
- There are algorithms for fixed recurrence operators (Beke 1894, Bronstein '94, Zhou–van Hoeij '19, ...)
  for computing factors of differen(tial/ce) operators.
- For Franel numbers, order 1 can be ruled out for all $s \geqslant 3$ (Yuan–Lu–Schmidt '08)
  using congruential properties.
- If $A(n+1)/A(n) \to \mu$ for $\mu \in \bar{\mathbb{Q}}$ of degree d, then $A(n)$ cannot satisfy a recurrence over $\mathbb{Q}$ of order less than d. (McIntosh '89)

  For Apéry numbers: $\mu = (1+\sqrt{2})^4$.
  For Franel numbers: $\mu = 2^s$. Not helpful!

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n}\binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n}\binom{n}{k}^s \left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s} = \sum_{j\geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n}\binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n}\binom{n}{k}^s \left[\prod_{j=1}^{k}\left(1-\frac{t}{j}\right)\prod_{j=1}^{n-k}\left(1+\frac{t}{j}\right)\right]^{-s} = \sum_{j\geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $\lim_{n\to\infty}\frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}]\left(\frac{\pi t}{\sin(\pi t)}\right)^s \in \pi^{2j}\mathbb{Q}_{>0}$

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)}\right)^s \in \pi^{2j}\mathbb{Q}_{>0}$

- Our proof is based on showing locally uniform convergence in t of

$$\lim_{n\to\infty} \frac{A^{(s)}(n,t)}{\sum_{k=0}^{n} \binom{n}{k}^s} = \left(\frac{\pi t}{\sin(\pi t)}\right)^s.$$

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^{s}$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.
(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^{s} \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $\lim_{n \to \infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^{s} \in \pi^{2j} \mathbb{Q}_{>0}$

- Our proof is based on showing locally uniform convergence in t of

$$\lim_{n \to \infty} \frac{A^{(s)}(n,t)}{\sum_{k=0}^{n} \binom{n}{k}^{s}} = \left(\frac{\pi t}{\sin(\pi t)} \right)^{s}.$$

"poof" For large n and $k \approx n/2$,

$$\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \approx \prod_{j=1}^{\infty} \left(1 - \frac{t}{j}\right) \left(1 + \frac{t}{j}\right) = \frac{\sin(\pi t)}{\pi t}.$$

□

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.
(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j} \mathbb{Q}_{>0}$

- In the case $j = 1$, this settles previous conjectures:

Apéry limits and lower bounds

THM S-Zudilin '21

Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.

(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21

$$\lim_{n \to \infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j} \mathbb{Q}_{>0}$$

- In the case $j = 1$, this settles previous conjectures:
  - $s = 3, 4$ numerically observed by Cusick (1979)
  - $s = 3$ proved by Zagier (2009)
  - $s = 5$ conjectured by Almkvist, van Straten, Zudilin (2008)
  - $s \geqslant 3$ conjectured by Chamberland–S (2020)

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.
(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $$\lim_{n \to \infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j} \mathbb{Q}_{>0}$$

- In the case $j = 1$, this settles previous conjectures:
  - $s = 3, 4$ numerically observed by Cusick (1979)
  - $s = 3$ proved by Zagier (2009)
  - $s = 5$ conjectured by Almkvist, van Straten, Zudilin (2008)
  - $s \geqslant 3$ conjectured by Chamberland–S (2020)

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ has order at least $\lfloor \frac{s+1}{2} \rfloor$.

Apéry limits and lower bounds

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ solved by $A_j^{(s)}(n)$ if $0 \leqslant 2j < s$.
(fine print: for large enough n)

$$A^{(s)}(n,t) := \sum_{k=0}^{n} \binom{n}{k}^s \left[\prod_{j=1}^{k} \left(1 - \frac{t}{j}\right) \prod_{j=1}^{n-k} \left(1 + \frac{t}{j}\right) \right]^{-s} = \sum_{j \geqslant 0} A_j^{(s)}(n)\, t^{2j}$$

THM S-Zudilin '21 $\lim_{n\to\infty} \frac{A_j^{(s)}(n)}{A^{(s)}(n)} = [t^{2j}] \left(\frac{\pi t}{\sin(\pi t)} \right)^s \in \pi^{2j} \mathbb{Q}_{>0}$

- In the case $j = 1$, this settles previous conjectures:
  - $s = 3, 4$ numerically observed by Cusick (1979)
  - $s = 3$ proved by Zagier (2009)
  - $s = 5$ conjectured by Almkvist, van Straten, Zudilin (2008)
  - $s \geqslant 3$ conjectured by Chamberland–S (2020)

THM S-Zudilin '21 Any telescoping recurrence for $\sum_{k=0}^{n} \binom{n}{k}^s$ has order at least $\lfloor \frac{s+1}{2} \rfloor$.

- This implies Franel's conjecture on the exact order
  if the minimal-order recurrence is telescoping. True at least for $s \leqslant 30$.

Conclusions and some open questions

- Applications of Apéry limits:
  - Irrationality proofs for $\zeta(2)$ and $\zeta(3)$
  - Explicitly construct the solutions guaranteed by Perron's theorem
  - Continued fractions
  - Prove lower bounds on orders of recurrences **new!**

Conclusions and some open questions

- Applications of Apéry limits:
  - Irrationality proofs for $\zeta(2)$ and $\zeta(3)$
  - Explicitly construct the solutions guaranteed by Perron's theorem
  - Continued fractions
  - Prove lower bounds on orders of recurrences **new!**

- Many open questions! For instance:
  - Cusick '89 and Stoll '97 construct recurrences for Franel numbers. Can these constructions produce telescoping recurrences?
  - What can we learn from other families of binomial sums?

    Also, it would be nice to simplify some of the technical steps in the arguments.
  - Can we (uniformly) establish the conjectural Apéry limits for CY DE's?
  - Can we explain when CT falls short? And algorithmically "fix" this issue?

THANK YOU!

Slides for this talk will be available from my website:
http://arminstraub.com/talks

M. Chamberland, A. Straub
Apéry limits: Experiments and proofs
American Mathematical Monthly, Vol. 128, Nr. 9, 2021, p. 811-824

A. Straub, W. Zudilin
Sums of powers of binomials, their Apéry limits, and Franel's suspicions
International Mathematics Research Notices, to appear, 2022. arXiv:2112.09576