An invitation to constant term sequences

Clifford Lectures 2024
The Web of Modularity

Tulane University — February 22–25, 2024

Armin Straub

February 25, 2024

University of South Alabama

$$\operatorname{ct}[f(x)] = \frac{1}{2\pi i}\int_{|x|=\varepsilon} f(x)\frac{\mathrm{d}x}{x}$$

$$\frac{1}{n+1}\binom{2n}{n} = \operatorname{ct}\left[\left(\frac{1}{x}+2+x\right)^n (1-x)\right]$$

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \operatorname{ct}\left[\left(\frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz}\right)^n\right]$$

Slides available at:
`http://arminstraub.com/talks`

based on joint work with:

Alin Bostan
(Université Paris-Saclay)

Sergey Yurkevich
(University of Vienna)

- $c(n)$ is D-**finite** if it satisfies a linear recurrence with polynomial coefficients.

EG The **Apéry numbers** $A(n)$ satisfy $A(0) = 1$, $A(1) = 5$ and

$$(n+1)^3 A(n+1) = (2n+1)(17n^2+17n+5)A(n) - n^3 A(n-1).$$

$\zeta(3)$ is irrational!

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CONJ Christol '90 Every D-finite integer sequence with at most exponential growth is the diagonal of a rational function.

EG S 2014 The Apéry numbers are the diagonal of $\frac{1}{(1-x-y)(1-z-w)-xyzw}$.

Diagonals

$$\sum_{n_1,\ldots,n_d \geqslant 0} a(n_1,\ldots,n_d)\, x_1^{n_1} \cdots x_d^{n_d}$$

multivariate series

$$\sum_{n \geqslant 0} a(n,\ldots,n)\, t^n$$

diagonal

EG

$$\frac{1}{1-x-y}$$

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$$\frac{1}{1-x-y} = \sum_{k=0}^{\infty} (x+y)^k = \sum_{n,m \geqslant 0} \binom{m+n}{m} x^m y^n$$

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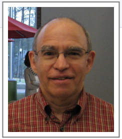
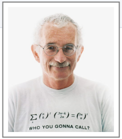
THM Gessel, Zeilberger, Lipshitz 1981–88

The diagonal of a rational function is D-finite.

More generally, the diagonal of a D-finite function is D-finite.

$F \in K[[x_1,\ldots,x_d]]$ is D-finite if its partial derivatives span a finite-dimensional vector space over $K(x_1,\ldots,x_d)$.





Characterizations of diagonals

EG Diagonals of rational functions

- $F(x) \quad = \quad C$-finite sequences

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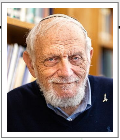
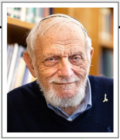


EG Diagonals of rational functions

- $F(x)$ = C-finite sequences
- $F(x,y)$ = sequences with algebraic GF (Furstenberg '67)

To see the latter, express the diagonal as $\frac{1}{2\pi i}\int_{|x|=\varepsilon} F\left(x, \frac{z}{x}\right)\frac{\mathrm{d}x}{x}$.



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THM Bostan, Lairez, Salvy '17: Diagonals of rational functions = (multiple) binomial sums

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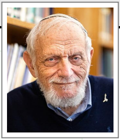

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CONJ Christol '90: Diagonals of rational functions over $\mathbb{Q}$ ($\subseteq$ known)

$=$ globally bounded, D-finite sequences

(i.e. $cd^n a_n \in \mathbb{Z}$ for $c, d \in \mathbb{Z}$ and at most exponential growth)

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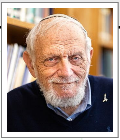

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- Open: example of a diagonal that requires more than 3 variables

Though we have numerous candidates.

The Apéry numbers as diagonals

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S 2014

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- Well-developed theory of multivariate asymptotics — e.g., Pemantle–Wilson
- OGFs of such diagonals are algebraic modulo p^r. — Furstenberg, Deligne '67, '84

  Automatically leads to congruences such as

$$A(n) \equiv \begin{cases} 1 \pmod{8}, & \text{if } n \text{ even}, \\ 5 \pmod{8}, & \text{if } n \text{ odd}. \end{cases}$$

Chowla–Cowles–Cowles '80
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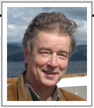
- Univariate generating function:

$$\sum_{n\geqslant 0} A(n)x^n = \frac{17-x-z}{4\sqrt{2}(1+x+z)^{3/2}} \, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| -\frac{1024x}{(1-x+z)^4}\right),$$

where $z=\sqrt{1-34x+x^2}$.

Strands of the web of modularity



THM Beukers '87

$$\underbrace{\frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)}}_{\text{modular form}} = \sum_{n\geqslant 0} A(n) \underbrace{\left(\frac{\eta^{12}(\tau)\eta^{12}(6\tau)}{\eta^{12}(2\tau)\eta^{12}(3\tau)}\right)^n}_{\text{modular function}}$$

$1 + 5q + 13q^2 + 23q^3 + O(q^4)$ $\quad$ $q - 12q^2 + 66q^3 + O(q^4)$ $\quad$ $q = e^{2\pi i\tau}$

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$$A(n) \equiv A(n_0)A(n_1)\cdots A(n_r) \pmod{p}$$

n_i are the p-adic digits of n

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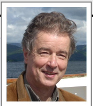

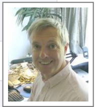

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THM Zagier '16

$$A\left(-\tfrac{1}{2}\right) = \frac{16}{\pi^2} L(f, 2)$$

- These extend to **all other** known Apéry-like numbers!!???

! = proven
? = partially known

Constant term representations

EG constant terms $A(n) = \operatorname{ct}[L^n]$ with $L = \dfrac{(1+y)(1+z)(1+x+z)(1+x+z+yz)}{xyz}$

- $F_A(t) = \sum_{n \geqslant 0} A(n) t^n = \operatorname{ct}\left[\dfrac{1}{1-tL}\right]$ is a **period function**.

  The DE satisfied by $F_A(t)$ is the **Picard–Fuchs DE** for the family $V_t : 1 - tL = 0$.

  Generically, V_t is birationally equivalent to a **K3 surface** with Picard number 19. (Beukers–Peters '84)

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THM Gorodetsky '21: Each sporadic sequence, except possibly (η), can be expressed as $\operatorname{ct}[P(\boldsymbol{x})^n]$ so that the result of Samol–van Straten applies.

EG Gorodetsky '21: (η): $\dfrac{(zx+xy-yz-x-1)(xy+yz-zx-y-1)(yz+zx-xy-z-1)}{xyz}$

$(1,0,0)$, $(1,1,0)$ and their permutations are interior points.

Constant terms

- $c(n)$ is a **constant term** if $c(n) = \mathrm{ct}[P^n(\boldsymbol{x})Q(\boldsymbol{x})]$ for Laurent polynomials $P, Q \in \mathbb{Q}[\boldsymbol{x}^{\pm 1}]$ in $\boldsymbol{x} = (x_1, \ldots, x_d)$.

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Q Zagier '16

Which integer sequences are constant terms?

And in which case can we choose $Q = 1$?

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- Constant terms are necessarily diagonals.

$$\frac{Q(\boldsymbol{x})}{1 - t x_1 \cdots x_d P(\boldsymbol{x})}$$

Q

Which diagonals are constant terms?

Which are linear combinations of constant terms?

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- We will answer this in the case of a single variable. (C-finite sequences!)
- For instance: Are Fibonacci numbers constant terms? $\dfrac{x}{1-x-x^2}$

Are Fibonacci numbers constant terms?

- Our key ingredient to answer these questions are **congruences**:

LEM Bostan, S, Yurkevich '23

If $A(n)$ is a constant term then, for all large enough primes p,

$$A(p) \equiv \underset{\in \mathbb{Q}}{\text{const}} \pmod{p}.$$

proof $A(p) = \operatorname{ct}[P(\boldsymbol{x})^p Q(\boldsymbol{x})] \equiv \operatorname{ct}[P(\boldsymbol{x}^p) Q(\boldsymbol{x})] = Q(\boldsymbol{0}) \operatorname{ct}[P(\boldsymbol{x}^p)] = Q(\boldsymbol{0}) \operatorname{ct}[P(\boldsymbol{x})]$ □

EG The Fibonacci numbers are $F(n) = \dfrac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}$ with $\varphi_\pm = \dfrac{1 \pm \sqrt{5}}{2}$.

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The Fibonacci numbers are $F(n) = \dfrac{\varphi_+^n - \varphi_-^n}{\sqrt{5}}$ with $\varphi_\pm = \dfrac{1 \pm \sqrt{5}}{2}$.

It follows that

$$F(p) \equiv \begin{cases} 1, & \text{if } p \equiv 1, 4 \bmod 5, \\ -1, & \text{if } p \equiv 2, 3 \bmod 5, \end{cases} \pmod{p}.$$

Hence, the Fibonacci numbers cannot be constant terms.

C-finite sequences that are constant terms

THM Bostan, S, Yurkevich '23

A C-finite sequence $A(n)$ is an r-term $\mathbb{Q}$-linear combination of constant terms if and only if it has at most r distinct characteristic roots, all of which are rational.

If the $A(n)$ are integers, then the characteristic roots are integers. (Carlo Sanna '23)

EG The only C-finite constant terms $A(n) = \mathrm{ct}[P(\boldsymbol{x})^n Q(\boldsymbol{x})]$ are:

- sequences with finite support (characteristic root 0),
- $\mathrm{poly}(n)\lambda^n, \quad \lambda \in \mathbb{Q}$.

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EG $n^2\, 2^n$ is a constant term.

$$= \operatorname{ct}\left[(x+2)^n\left(\frac{8}{x^2} + \frac{2}{x}\right)\right]$$

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C-finite sequences that are constant terms

THM Bostan, S, Yurkevich '23

A C-finite sequence $A(n)$ is an r-term $\mathbb{Q}$-linear combination of constant terms if and only if it has at most r distinct characteristic roots, all of which are rational.

If the $A(n)$ are integers, then the characteristic roots are integers. (Carlo Sanna '23)

EG The only C-finite constant terms $A(n) = \operatorname{ct}[P(\boldsymbol{x})^n Q(\boldsymbol{x})]$ are:

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EG Fibonacci and Lucas numbers are not (sums of) constant terms.

The case of pure powers

Q Zagier '16

Which integer sequences are constant terms $\operatorname{ct}[P(\boldsymbol{x})^n]$?

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   (**Gauss congruences**)

- Gauss congruences satisfied by **realizable** sequences $a(n)$:

$$a(n) = \#\{x \in X : T^n x = x\}$$

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Everest–van der Poorten–Puri–Ward '02, Arias de Reyna '05

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THM Minton, 2014 — Let $A(n)$ be C-finite. TFAE:

1. $A(n)$ is a trace sequence.
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- A sequence $c(n)$ is **hypergeometric** if $\frac{c(n+1)}{c(n)}$ is a rational function.

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EG open!: Is the following hypergeometric sequence a diagonal?

$$A(n) = \frac{\left(\frac{1}{9}\right)_n \left(\frac{4}{9}\right)_n \left(\frac{5}{9}\right)_n}{n!^2 \left(\frac{1}{3}\right)_n}$$

$3^{6n} A(n) = 1, 60, 20475, 9373650, 4881796920, \ldots$

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LEM Bostan, S, Yurkevich '23 — This hypergeometric sequence is not a constant term (or a linear combination of constant terms).

Proof idea: $A(p)$ takes different values modulo p depending on whether $p \equiv \pm 1 \pmod 9$.

Constant terms are special

- For hypergeometric sequences: (or C-finite or D-finite sequences)

$$\underset{\text{(or linear combinations)}}{\{\text{constant terms}\}} \subsetneq \{\text{diagonals}\} \subseteq \{P\text{-finite \& globally bounded seq's}\}$$

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- The following is an indication that constant terms are special among diagonals and often have significant additional arithmetic properties.

LEM Bostan, S, Yurkevich '23

Let $A_m(n) = \frac{\left(\frac{1}{m}\right)_n \left(1 - \frac{1}{m}\right)_n}{n!^2}$ where $m \geqslant 2$ is an integer.

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1. $A_m(n)$ is a diagonal for all $m \geqslant 2$.
2. $A_m(n)$ is a constant term if and only if $m \in \{2, 3, 4, 6\}$.

- The cases $m \in \{2, 3, 4, 6\}$ correspond to the hypergeometric functions underlying Ramanujan's theory of elliptic functions.
  ($m = 2$: classical case; $m = 3, 4, 6$: alternative bases)

Conclusions & Outlook

- Constant terms are an arithmetically interesting subset of diagonals.
- We have classified them in the case of a single variable. Natural classes of sequences to consider next:
  - Hypergeometric sequences
  - Algebraic sequences (diagonals in two variables)
  - Algebraic hypergeometric series
  - Integral factorial ratios (Bober, 2007; via Beukers–Heckman)

EG Is $A(n) = \frac{(8n)!n!}{(4n)!(3n)!(2n)!} = \binom{8n}{4n}\binom{4n}{n}\binom{2n}{n}^{-1}$ a constant term?

$$1, 140, 60060, 29745716, 15628090140, \ldots = \operatorname{ct}\left[\left(\frac{(1+x)^8}{(1-x)^2x^3}\right)^n\right]$$

This is algebraic (and therefore a diagonal) and hypergeometric.

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This is algebraic (and therefore a diagonal) and hypergeometric.

- How to find representations as (nice) constant terms or diagonals?

  Once found, such representations can be proved using **creative telescoping**.
- How unique are the Laurent polynomials in a constant term?

  Connections to cluster algebras, mutations of Laurent polynomials, ...

THANK YOU!

Slides for this talk will be available from my website:
`http://arminstraub.com/talks`

A. Bostan, A. Straub, S. Yurkevich
On the representability of sequences as constant terms
Journal of Number Theory, Vol. 253, 2023, p. 235–256