Example 15. Let $f(x) = 1/x$. Determine $\lim_{x\to 2} f(x)$ and $\lim_{x\to 0} f(x)$. What about $\lim_{x\to 0^+} f(x)$?

Solution. Make a sketch!

- $\lim_{x\to 2} f(x) = f(2) = \frac{1}{2}$
- $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$ so that (because the two are different) $\lim_{x\to 0} f(x)$ does not exist.

If $f(x)$ is **continuous** at $x = c$, then $\lim_{x\to c} f(x) = f(c)$.

Actually, this will soon be our definition of what it means for a function to be continuous at a point.
For now, think of continuous at $x = c$ meaning that $f(x)$ does not have any sort of jump (or problem) at $x = c$. All of the basic functions we are familiar with are continuous everywhere except at points with an obvious problem (like a division by zero).

Example 16. Determine $\lim_{x\to 2} \frac{x^2 - x + 3}{x^2 - 1}$.

Solution. There is no problem at $x = 2$, so that we can just plug in $x = 2$: $\lim_{x\to 2} \frac{x^2 - x + 3}{x^2 - 1} = \frac{2^2 - 2 + 3}{2^2 - 1} = \frac{5}{3}$.

We have to work harder in cases, where there is an issue preventing us from just plugging in:

Example 17. Determine $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}$. [The denominator is 0 for $x = 2$, so we cannot just plug in.]

Note. Top and bottom both approach 0 as $x \to 2$, which means that the limit is of the undetermined form "$\frac{0}{0}$" and we need to work to determine whether the limit exists or not, and what its value might be.

Solution. Note that $\frac{x^2 - x - 2}{x^2 - 4} = \frac{(x-2)(x+1)}{(x-2)(x+2)} = \frac{x+1}{x+2}$.

Hence, $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4} = \lim_{x\to 2} \frac{x+1}{x+2} = \frac{3}{4}$.

Example 18. Determine $\lim_{x\to 2} \frac{\sqrt{x^2+5} - 3}{x - 2}$. [Again, note that we cannot just plug in.]

Solution. We need to use a standard "trick": multiplying both numerator and denominator with $\sqrt{x^2+5} + 3$.

$$\frac{\sqrt{x^2+5} - 3}{x - 2} = \frac{\left(\sqrt{x^2+5} - 3\right)\left(\sqrt{x^2+5} + 3\right)}{(x-2)\left(\sqrt{x^2+5} + 3\right)}$$
$$= \frac{(x^2+5) - 3^2}{(x-2)\left(\sqrt{x^2+5} + 3\right)} = \frac{x^2 - 4}{(x-2)\left(\sqrt{x^2+5} + 3\right)} = \frac{x+2}{\sqrt{x^2+5} + 3}$$

Hence, $\lim_{x\to 2} \frac{\sqrt{x^2+5} - 3}{x - 2} = \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+5} + 3} = \frac{2+2}{\sqrt{2^2+5} + 3} = \frac{2}{3}$.

Comment. Although we cannot plug in $x = 2$ into the original expression, we can plug in values close to 2 to get a numerical impression: for instance, for $x = 2.01$, we get $\frac{\sqrt{x^2+5} - 3}{x - 2} \approx 0.668$.

Example 19. Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for $f(x)=x^2+2$ and $x=3$.

Solution. First, $f(3)=11$ and $f(3+h)=(3+h)^2+2=h^2+6h+11$, so that

$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{(h^2+6h+11)-11}{h}=\lim_{h\to 0}\frac{h^2+6h}{h}=\lim_{h\to 0}(h+6)=6.$$

Comment. We could have done that computation without choosing a special value for x.
Since $f(x+h)=(x+h)^2+2=x^2+2hx+h^2+2$, we have

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x^2+2hx+h^2+2)-(x^2+2)}{h}=\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}(2x+h)=2x.$$

Advanced comment. This limit defines the **derivative** of $f(x)$, which we will meet later and denote $f'(x)$. In other words, our computation shows that $f(x)=x^2+2$ has derivative $f'(x)=2x$.
Can you see how the limit describes the slope of the line tangent to the graph of f at x?

(limit laws) If $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, then:

- $\lim_{x\to c}[f(x)+g(x)]=L+M$
- $\lim_{x\to c}[f(x)-g(x)]=L-M$
- $\lim_{x\to c}[f(x)g(x)]=LM$
- $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}$ (provided that $M\neq 0$)

It follows from the product rule that $\lim_{x\to c}[f(x)]^n=L^n$ for all positive integers n.

Similarly, we have $\lim_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{L}$.

Advanced comment. In fact, we will learn that $\lim_{x\to c} g(f(x))=g(L)$ provided that $g(x)$ is continuous at $x=L$.

Example 20. Suppose $\lim_{x\to a} f(x)=3$ and $\lim_{x\to a} g(x)=5$. Determine $\lim_{x\to a}[f(x)^2-7g(x)]$.

Solution. Using the limit laws,

$$\begin{aligned}\lim_{x\to a}[f(x)^2-7g(x)] &= \lim_{x\to a}[f(x)^2]-\lim_{x\to a}[7g(x)]\\ &= 3^2-7\cdot 5=-26.\end{aligned}$$

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