Continuity

(Definition of continuity) $f(x)$ is **continuous** at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

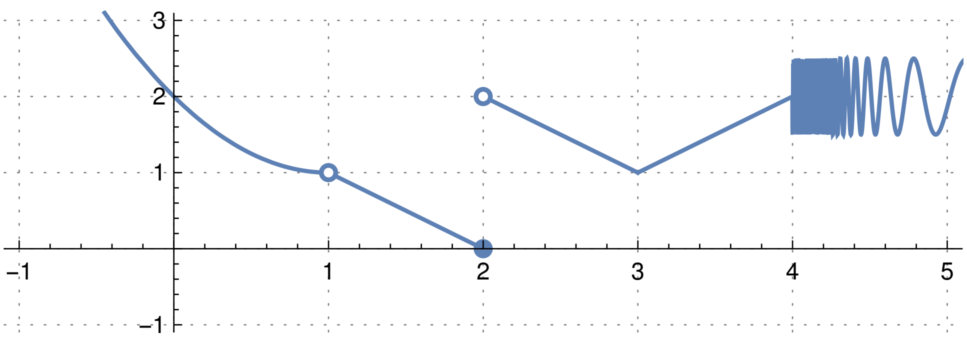
When we simply say that f is continuous, we mean that it is continuous at every point of its domain.

Note that, in particular, $f(x)$ needs to be defined at $x = c$.
The following are typical ways in which a function might fail to be continuous at a point:

- jump discontinuity,
- infinite discontinuity,
- removable discontinuity,
- oscillating discontinuity.

Make a sketch illustrating each case!

Example 25. Discuss where the sketched function is continuous and the kind of discontinuities.



Solution. The function $f(x)$ is continuous everywhere except at $x = 1$ (a removable discontinuity), $x = 2$ (a jump discontinuity) and $x = 4$ (an oscillating discontinuity).

Note that $x = 1$ is "removable" because we can define $f(1) = 1$ and this extended function would then be continuous at $x = 1$.

Good News! All the basic functions

$$\text{polynomials}, \quad \text{exponentials}, \quad \text{trig functions}, \quad |x|$$

(and all functions we can build through adding, subtracting, multiplying, dividing, compositions and inverse functions) are continuous at every point of their domain. More precisely:

- If f and g are both continuous at c, then

  $$f + g, \quad f - g, \quad fg, \quad \frac{f}{g}, \quad f^n, \quad \sqrt[n]{f}$$

  are each continuous at c.
- If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.
- If f is continuous (on all of its domain), then so is its inverse function f^{-1}.

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Note that the statement about continuity of compositions is equivalent to the following:

If $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} g(f(x)) = g(L)$ provided that g is continuous at L.

Example 26. Determine the points at which $f(x) = \cos(|2x+7|) - e^{\sin(7x-1)}$ is continuous.

Solution. Since $f(x)$ is constructed from functions which are continuous everywhere, we know that $f(x)$ is continuous at every point of its domain. Hence, $f(x)$ is continuous for all x.

Comment. Make sure you know what exactly is meant by "constructed" here. For instance $\cos(|2x+7|)$ is constructed as a composition of $\cos(x)$, $|x|$ and $2x+7$.

Example 27. Determine the points at which $f(x) = \frac{\cos(x)}{\sin(x)+1}$ is continuous.

Solution. Since $f(x)$ is constructed from functions which are continuous everywhere, we know that $f(x)$ is continuous at every point of its domain. Hence, $f(x)$ is continuous for all x for which $\sin(x) \neq -1$. That is, all x except the points $-\frac{\pi}{2} + 2\pi m$, where m is an integer.

Example 28. Define $f(1)$ in a way that extends $f(x) = \frac{x^2+2x-3}{x-1}$ to be continuous at $x=1$.

Solution. Note that $\frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{x-1} = x+3$ (for all $x \neq 1$).

We can therefore define $f(1) = 1+3 = 4$ to obtain a function $f(x)$ that is continuous everywhere.

Note. The resulting function is $f(x) = x+3$ (which makes this example a bit artificial). The next example illustrates that we usually cannot simply cancel terms and replace the function with the simplified expression.

Example 29. (a nontrivial continuous extension)

(a) Determine the points at which $f(x) = \frac{\sin(x)}{x}$ is continuous.

(b) If possible, extend $f(x)$ in such a way that it is continuous everywhere.

Solution.

(a) Since $f(x)$ is a quotient of functions ($\sin(x)$ and x) which are continuous everywhere, we know that $f(x)$ is continuous at every point of its domain. Hence, $f(x)$ is continuous for all $x \neq 0$.

(b) Recall that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

Hence, defining $f(0) = 1$ results in a function which is continuous everywhere.