Rules for computing derivatives

- $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ **(power rule)**
- $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$ **(derivative of e^x)**
- $\frac{\mathrm{d}}{\mathrm{d}x} [a f(x) + b g(x)] = a f'(x) + b g'(x)$ **(sum rule)**
- $\frac{\mathrm{d}}{\mathrm{d}x} [f(x) g(x)] = f'(x) g(x) + f(x) g'(x)$ **(product rule)**
- $\frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{f(x)}{g(x)} \right] = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}$ **(quotient rule)**

Why? We could derive the power rule via a computation similar to what we did in the last two classes. For the derivative of a^x, note that

$$\frac{a^{x+h} - a^x}{h} = a^x \frac{a^h - 1}{h} \xrightarrow{h \to 0} L a^x, \quad \text{where } L = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Euler's number $e \approx 2.718$ is defined precisely so that $L = 1$. [We will see shortly that, in general, $L = \ln(a)$.]
The sum rule follows from the corresponding limit rule:

$$\frac{[a f(x+h) + b g(x+h)] - [a f(x) + b g(x)]}{h} = a \frac{f(x+h) - f(x)}{h} + b \frac{g(x+h) - g(x)}{h} \xrightarrow{h \to 0} a f'(x) + b g'(x)$$

For the product rule:

$$\begin{aligned} \frac{f(x+h) g(x+h) - f(x) g(x)}{h} &= \frac{[f(x+h) g(x+h) - f(x) g(x+h)] + [f(x) g(x+h) - f(x) g(x)]}{h} \\ &= \frac{[f(x+h) - f(x)]}{h} g(x+h) + f(x) \frac{g(x+h) - g(x)}{h} \\ &\xrightarrow{h \to 0} f'(x) g(x) + f(x) g'(x) \end{aligned}$$

The quotient rule can be derived similarly but actually follows from the product rule and the chain rule (later!).

Example 56. What is $f'(x)$ in each case?

(a) $f(x) = x^2$

(b) $f(x) = x^4$

(c) $f(x) = 2^6$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = \frac{1}{x^2}$

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Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x^1 = 2x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^4 = 4x^3$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}2^6 = 0$ ($2^6 = 64$ is just a constant!)

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^2} = \frac{\mathrm{d}}{\mathrm{d}x}x^{-2} = -2x^{-3}$

Alternative solution via quotient rule. Write $f(x) = \frac{u(x)}{v(x)}$ with $u(x) = 1$ and $v(x) = x^2$.

$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{0 \cdot x^2 - 1 \cdot (2x)}{(x^2)^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$

Example 57. What is $f'(x)$ if $f(x) = -2x^4 + 3x^5$?

Solution. $f'(x) = -2(4x^3) + 3(5x^4) = -8x^3 + 15x^4$

(higher order derivatives) The derivative of the derivative is the **second derivative**.
It is denoted $f''(x)$ or $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)$. Or, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$.

Similarly, but less important, there is a third derivative, and so on...

Example 58. Compute the first six derivatives of $f(x) = -2x^4 + 3x^5$.

Solution.

- $f'(x) = -8x^3 + 15x^4$
- $f''(x) = -24x^2 + 60x^3$
- $f'''(x) = -48x + 180x^2$
- $f^{(4)}(x) = -48 + 360x$
- $f^{(5)}(x) = 360$
- $f^{(6)}(x) = 0$

All further derivatives will be zero.

Comment. If $f(x)$ is a polynomial of degree n (here, $n = 5$), then $f'(x)$ is a polynomial of degree $n - 1$.

Example 59. Compute the derivatives of the following functions.

(a) $h(x) = (3x^2 - 1)e^x$

(b) $h(x) = \frac{x^2 - 2}{3x + 7}$

Solution.

(a) Write $h(x) = f(x)g(x)$ with $f(x) = 3x^2 - 1$ and $g(x) = e^x$.
$h'(x) = f'(x)g(x) + f(x)g'(x) = (6x)e^x + (3x^2 - 1)e^x = (3x^2 + 6x - 1)e^x$

(b) Write $h(x) = \frac{f(x)}{g(x)}$ with $f(x) = x^2 - 2$ and $g(x) = 3x + 7$.

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{(2x) \cdot (3x + 7) - (x^2 - 2) \cdot 3}{(3x + 7)^2} = \frac{3x^2 + 14x + 6}{(3x + 7)^2}$