Applications of derivatives

Extreme values

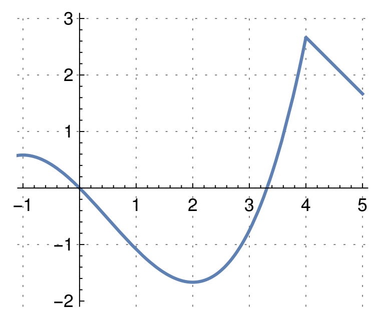
Theorem 85. (extreme value theorem) A function that is continuous on a closed interval $[a, b]$ attains both an absolute maximum value and an absolute minimum value on $[a, b]$.

Important. Via a sketch (as we did in class), demonstrate how the extreme value theorem fails in the following instances:

- if we drop the assumption of continuity (even if the function is bounded)
- if $[a, b]$ is replaced with, say, $(a, b]$ (again, even if the function is bounded)

Example 86. Sketch a continuous function $f(x)$ with the property that $f'(1) = -1$, $f'(2) = 0$, $f'(3) = 2$, and so that $f'(4)$ does not exist.

Solution. The graph to the right is an example of such a function.



Theorem 87. (f' at local extrema) Let c be a point in, but not on the boundary of, the domain of f. If f has a local extremum at c, and $f'(c)$ is defined, then $f'(c) = 0$.

Crucial. This provides a means to find local extrema! Namely, these can only occur:

(a) at the boundary of the domain,

(b) at values c such that $f'(c) = 0$, or

(c) at values c such that $f'(c)$ is not defined.

(critical point) An interior point c of the domain of f is a **critical point** if either $f'(c) = 0$ or $f'(c)$ is undefined.

It follows that extreme values (both local and absolute) can only occur on the boundary of the domain or at critical points.

(recipe for finding absolute extrema) To find the absolute minimum and absolute maximum of a continuous function $f(x)$ on $[a, b]$:

(a) evaluate f at the endpoints (a and b) and all critical points, and

(b) take the largest and smallest of these.

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Example 88. Consider $f(x) = xe^x$.

(a) Find all critical points of $f(x)$.

(b) Then find all extrema (both local and absolute) on the interval $[-2, 2]$.

Solution.

(a) We compute $f'(x) = 1 \cdot e^x + x \cdot e^x = (x+1)e^x$.
Solving $f'(x) = 0$, we find that the only critical point is at $x = -1$.

(b) The extreme values can only occur at -1 (critical point) or at -2, 2 (endpoints).

x	-2		-1		2
$f(x)$	$-2e^{-2} \approx -0.271$	↘	$-e^{-1} \approx -0.368$	↗	$2e^2 \approx 14.8$
$f'(x)$		$-$	0	$+$	

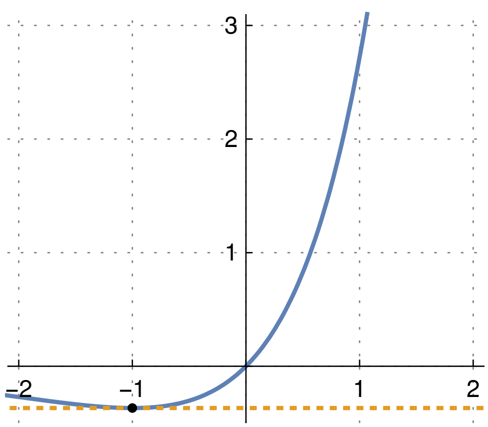
We conclude that f on $[-2, 2]$ has the following extrema:

- An absolute minimum of $-\frac{1}{e}$ at $x = -1$ and an absolute maximum of $2e^2$ at $x = 2$.
- An additional local maximum of $-2e^{-2}$ at $x = -2$.

Important comment. Between the values in our table, we indicated that f is decreasing (↘) on the interval $(-2, -1)$ and increasing (↗) on the interval $(-1, 2)$. Can you explain why that must be the case?

The derivative f' is itself continuous, so it can change from positive $(+)$ to negative $(-)$ only if it is zero in between (by the intermediate value theorem applied to f'). But we have already listed all points where $f' = 0$. Hence, on each interval between listed points, the derivative must be all positive (f increasing) or negative (f decreasing).

A plot that confirms our findings.



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