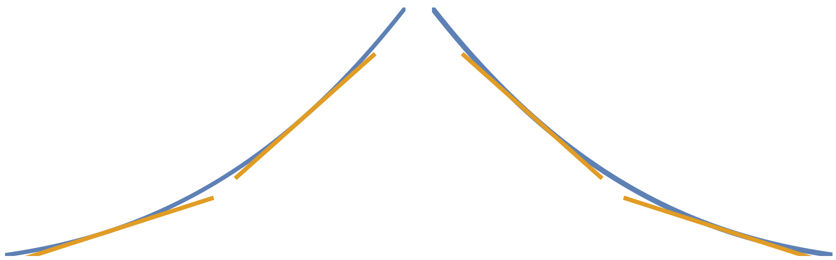
The second-derivative test

f is **concave up** on an open interval I.
$\Longleftrightarrow$ f' is increasing on I.

Graphically. This means that the graph of f lies above the tangent lines.



Being **concave down** is defined analogously.

An **inflection point** is a point where concavity is changing.
[From concave up to down, or the other way around.]
Equivalently, an inflection point is a local extremum of the derivative.

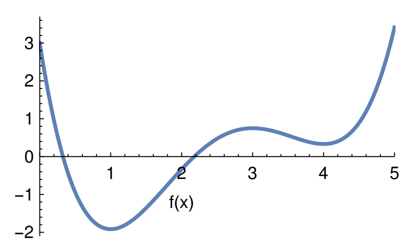
Intuitively. Here is a visual way to think of concavity and inflection points: Imagine yourself riding a bike along the graph of $f(x)$. If the graph is a straight line, then you are steering neither left nor right. Usually, however, the graph is curved and you will have to steer either a little left or a little right.
Steering left means the graph is concave up, steering right means the graph is concave down. An inflection point is a point where you are transitioning from steering one direction to the other.

(second-derivative rule)

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If $f''(a) = 0$, then f might have an inflection point at $x = a$.

Why? Recall that, if $g' > 0$ on I, then g is **increasing** on I. Apply that with $g = f'$.

Example 92. Let $f(x)$ be as sketched.



(a) On which intervals is $f(x)$ increasing/decreasing?

(b) Describe the slopes between $x = 1$ and $x = 3$.

(c) Sketch $f'(x)$.

(d) Approximately, where is $f(x)$ concave up/down?

(e) Approximately, what are the inflection points?

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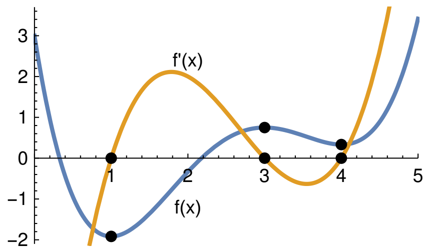
Solution.

(a) $f(x)$ is increasing on the open intervals $(1,3)$ and $(4,\infty)$. $f(x)$ is decreasing on $(-\infty,1)$ and $(3,4)$.

(b) For $1<x<3$, the slopes are positive (i.e. $f(x)$ is increasing).
But we can say more:
The slopes are increasing from $x=1$ until $x\approx 1.8$ (the maximal slope is about 2.1), then the slopes are descreasing from $x\approx 1.8$ to $x=3$.
The point $x\approx 1.8$ is special. It is an **inflection point**.

(c) A sketch of $f'(x)$:



(d) $f(x)$ is concave up for: $x<1.8$ and $x>3.5$
$f(x)$ is concave down for: $1.8<x<3.5$

(e) $f(x)$ has inflection points at: $x\approx 1.8$ and $x\approx 3.5$

(second-derivative test) Suppose $f'(a)=0$ and that $f''(a)$ exists.

- If $f''(a)<0$, then $f(x)$ has a local maximum at $x=a$.
- If $f''(a)>0$, then $f(x)$ has a local minimum at $x=a$.
- If $f''(a)=0$, then we don't know whether or not there is a local extremum at $x=a$.

Example 93. Find the local extrema of $f(x)=\frac{1}{3}x^3-2x^2+3x-1$.

Solution. (using second-derivative test)
We have $f'(x)=x^2-4x+3$. Solving $f'(x)=0$ we find that the critical points are $x=1$ and $x=3$.
Next, we compute the second derivative: $f''(x)=2x-4$

- $f''(1)=-2<0$ implies that f is concave down at $x=1$. That means f lies below the horizontal tangent line. We therefore must have a local maximum at $x=1$.
- $f''(3)=2>0$ implies that f is concave up at $x=3$. That means f lies above the horizontal tangent line. We therefore must have a local minimum at $x=3$.

Solution. (using first-derivative test, as in previous examples)
Again, we start with $f'(x)=x^2-4x+3$. Solving $f'(x)=0$ we find that the critical points are $x=1$ and $x=3$.

intervals	$x<1$	$x=1$	$1<x<3$	$x=3$	$x>3$	
$f'(x)$	$+$	0	$-$	0	$+$	we can determine the sign by computing $f'(x)$ for some x in the interval
$f(x)$	↗		↘		↗	

Hence, $f(x)$ has a local maximum at $x=1$ and a local minimum at $x=3$.

Example 94. (cont'd) Find the inflection points of $f(x)=\frac{1}{3}x^3-2x^2+3x-1$.

Solution. An inflection point is a local extremum of $f'(x)=x^2-4x+3$. Since $f''(x)=2x-4$, solving $f''(x)=0$, we find that there might be an inflection point at $x=2$. It is indeed an inflection point because f'' is changing from <0 (f being concave down) to >0 (f being concave up) at $x=2$.

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When to use which test?

Rule of thumb: If $f''(a)$ is easy to compute, use the second-derivative test.

Otherwise, or if $f''(a)=0$, use the first-derivative test.

[If we have computed all a such that $f'(x)=0$, then the first-derivative test is easy to apply because we can quickly determine the sign of $f'(x)$ for any x.]

Example 95. (again, with focus on f'') Consider $f(x)=\frac{x+1}{x^2+x+4}$.

(a) Find all critical points of f. For each critical point, use f'' to determine whether it is a local minimum or maximum.

(b) Sketch the function and indicate the inflection points. How would we compute those?

Solution.

(a) As last time, $f'(x)=\frac{-(x-1)(x+3)}{(x^2+x+4)^2}$. Solving $f'(x)=0$, we find that the critical points are 1 and -3.

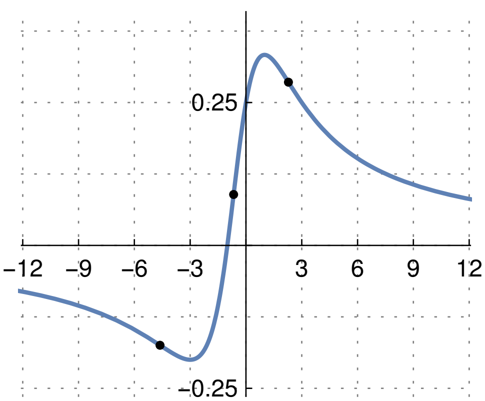
We further compute $f''(x)=\frac{2(x^3+3x^2-9x-7)}{(x^2+x+4)^3}$ and evaluate to find $f''(-3)=\frac{1}{25}$ and $f''(1)=-\frac{1}{9}$.

- $f''(-3)>0$ implies that f is concave up at $x=-3$. That means f lies above the horizontal tangent line. We therefore must have a local minimum at $x=-3$.
- $f''(1)<0$ implies that f is concave down at $x=1$. That means f lies below the horizontal tangent line. We therefore must have a local maximum at $x=1$.

Comment. Combined with the fact that $y=0$ is a horizontal asymptote of $f(x)$, both as $x\to\infty$ and $x\to-\infty$, we can conclude that both of these extrema have to be absolute extrema.

Comment. Here, computing f'' is actually a bit of work. If we had a choice, we would probably prefer to use the first-derivative test as we did earlier.

(b) The inflection points are indicated on the following sketch:



To compute these, we would solve $f''(x)=0$, which is equivalent to the equation $x^3+3x^2-9x-7=0$, which has three solutions: $x\approx-4.62$, $x\approx-0.663$, $x\approx2.28$ (but there is no particularly simple formula for these values). These are the x-coordinates of the inflection points indicated on the plot.

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