Review: Our zoo of functions

- polynomials
  x^2, x^3, $7x^4 - x + 2$, ...
- rational functions
  $\frac{1}{x+1}$, $\frac{x^2 - 2x - 3}{x^3 + 7}$, ...
- power functions
  x^2, $x^{1/2} = \sqrt{x}$, $x^{-1/2} = \frac{1}{\sqrt{x}}$, ...
- exponentials
  2^x, e^x, ...
- logarithms
  $\ln(x) = \log_e(x)$, $\log_2(x)$, ...
- trigonometric functions
  $\sin(x)$, $\cos(x)$, $\tan(x) = \frac{\sin(x)}{\cos(x)}$, ...
- inverse trig functions
  $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, ...

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x) g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2}.$$

Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x} (5x^3 + 7x^2 + 2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(5x^3 + 7x^2 + 2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} (x^3 + 2x) \sin(5x^3 + 7x^2 + 2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x} (5x^3 + 7x^2 + 2) = 15x^2 + 14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} \sin(5x^3 + 7x^2 + 2) = (15x^2 + 14x) \cos(5x^3 + 7x^2 + 2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} (x^3 + 2x) \sin(5x^3 + 7x^2 + 2)$
$= (3x^2 + 2) \sin(5x^3 + 7x^2 + 2) + (x^3 + 2x)(15x^2 + 14x) \cos(5x^3 + 7x^2 + 2)$

Example 3. Find $\frac{\mathrm{d}}{\mathrm{d}x} \tan(x)$ using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and the quotient rule.

Example 4. What is $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x)$?

Why? Can you explain why this is the case?
(One way to see this is to recall that $e^{\ln(x)} = x$ and to differentiate both sides. What do you conclude?)

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Review: Integration and areas

If $f(x) \geqslant 0$ for $x \in [a, b]$, then

$$\int_a^b f(x)\mathrm{d}x$$

is defined as the area enclosed by the graph of $f(x)$ and the x-axis between $x = a$ and $x = b$.

Can you explain how the integral $\int_a^b f(x)\,\mathrm{d}x$ is constructed from sums $\sum f(x)\Delta x$?
(Here, we are summing over rectangles of width Δx between a and b; at position x their height is roughly $f(x)$.)

Comment. Σ is a capital sigma, and just means "sum". Don't worry about it for now. We will see it again later.

Review: Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus connects the two operations

(a) differentiation and

(b) integration,

which, at first glance, look like they might be of a rather different nature. Roughly, it shows that these two operations are inverses of each other.

Theorem 5. (Fundamental Theorem of Calculus, part 2)

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

if $F(x)$ is an antiderivative of $f(x)$.

The "first part" of the Fundamental Theorem of Calculus is the statement that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

Modulo details (such as whether $f(x)$ is continuous or differentiable), can you conclude this "first part" from the "second part" above?
(Hint: in the "second part", replace b by x and differentiate both sides with respect to x.)

Example 6. Compute $\int_1^2 x\,\mathrm{d}x$ in two ways: first, making a sketch and using the definition as an area; then, using the Fundamental Theorem of Calculus.

Solution.

(a) Make a sketch! The area in question consists of a 1×1 square (area 1) with exactly half a 1×1 square (area $1/2$) on top. Hence, the total area is $1 + \frac{1}{2} = \frac{3}{2}$.

(b) $\int_1^2 x\,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_1^2 = \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot 1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

Example 7. $\int_0^\pi \sin(x)\,\mathrm{d}x =$

Why is $\int_0^{2\pi} \sin(x)\,\mathrm{d}x = 0$? Explain geometrically in terms of areas.

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Review: Antiderivatives (or indefinite integrals)

Definition 8. If $f(x) = F'(x)$ then we say that $F(x)$ is an **antiderivative** of $f(x)$.
We write: $\int f(x)\mathrm{d}x = F(x) + C$

This is also called the **indefinite integral** of $f(x)$.

Comment. The notation using the integral sign makes sense because of the Fundamental Theorem of Calculus.

Example 9. $\int x\,\mathrm{d}x = \frac{1}{2}x^2 + C$

Example 10. $\int x^a\,\mathrm{d}x = \frac{1}{a+1}x^{a+1} + C$

Comment. Note that the case $a = -1$ is special. What happens in that case?

Example 11. $\int \frac{1-x}{x^3}\,\mathrm{d}x = \int (x^{-3} - x^{-2})\mathrm{d}x = -\frac{1}{2}x^{-2} + x^{-1} + C$

Example 12. $\int x\sqrt{x}\,\mathrm{d}x = \int x^{3/2}\mathrm{d}x = \frac{2}{5}x^{5/2} + C$

Substitution

The following is a first example for which the antiderivative is not so readily obtained by reversing basic rules of differentiation.

Example 13. Determine $\int x\sqrt{x^2+1}\mathrm{d}x$ by substituting $u = x^2 + 1$.

Solution. We need to substitute all occurences of x in the integral, including the $\mathrm{d}x$.
To substitute the latter, note that if $u = x^2 + 1$ then the derivative with respect to x is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x.$$

Solving for $\mathrm{d}x$, we find $\mathrm{d}x = \frac{1}{2x}\mathrm{d}u$.
Substituting in the integral, we therefore find

$$\int x\sqrt{x^2+1}\mathrm{d}x = \int x\sqrt{u}\,\frac{1}{2x}\mathrm{d}u = \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C.$$

Note how in the final step, we substituted back $u = x^2 + 1$ to get the desired antiderivative in terms of x.

Comment. We could have been slightly more efficient by directly substituting $x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.

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