Review. The (arc) length of a general curve $y = f(x)$, for $a \leqslant x \leqslant b$, is given by

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x.$$

Example 42. (extra) Using our new technology, compute the circumference of a circle of radius r.

Solution. Of course, the final answer has to be $2\pi r$.

- First, in order to work with a function, we consider the (upper) half circle.

  That half-circle is described by $y = \sqrt{r^2 - x^2}$ and the length of that curve (from $x = -r$ to $x = r$) is half of the circumference of our circle.

- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{\sqrt{r^2 - x^2}}$

- Hence, the circumference of our circle is given by the integral

$$2\int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, \mathrm{d}x = 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, \mathrm{d}x = 2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, \mathrm{d}x.$$

- Now, we substitute $u = x/r$.

  Why? Just looking at the integral, the reason for choosing this substitution might not be obvious. However, thinking about our actual problem, this substitution is very natural: it scales things by $1/r$ so that our circle gets rescaled to a circle of radius 1.

It is very common in applications that we need to change scales or coordinate systems. When dealing with integrals, we then need to perform the corresponding substitution.

Such changes of scale or coordinate systems for practical reasons is a second important reason why we need to be able to substitute. (So far we substituted as a means to mathematically simplify an integral.)

- To substitute, we compute $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{r}$, and so $\mathrm{d}x = r\,\mathrm{d}u$ [r is just a number—we can treat it like we would treat 7.]

  For the boundaries of the integral: if $x = -r$, then $u = -1$. If $x = r$, then $u = 1$.

  Since $x = ru$, we therefore get

$$2\int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\, \mathrm{d}x = 2\int_{-1}^{1} \sqrt{\frac{r^2}{r^2 - (ru)^2}} \cdot r\,\mathrm{d}u = 2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\, \mathrm{d}u.$$

- We can actually evaluate this final integral (more on such integrals later) if we recall that the derivative of $\arcsin(u)$ is $1/\sqrt{1 - u^2}$. Hence,

$$2r\int_{-1}^{1} \frac{1}{\sqrt{1 - u^2}}\, \mathrm{d}u = 2r\Big[\arcsin(u)\Big]_{-1}^{1} = 2r[\arcsin(1) - \arcsin(-1)] = 2\pi r.$$

Physical work

Work is **force** times **distance**: $W = Fd$.

- F could be measured in lb and d in ft. Then W is conveniently measured in $\mathrm{ft\text{-}lb}$.
- The SI units for F are N (newton), for d they are m (meter), and W is measured in Nm (newtonmeter) or joule ($1\ \mathrm{Nm} = 1\ \mathrm{joule} \approx 0.738\ \mathrm{ft\text{-}lb}$).

Think about the work necessary to lift an object to a certain height.

Calculus comes in when the force F is not constant!

Example 43. Suppose we wish to lift a $100\ \mathrm{lb}$ piano from the ground to the top of a $20\ \mathrm{ft}$ building (for instance, by standing on the roof and pulling it up using a rope). The work required for that is

$$\mathrm{work} = (100\ \mathrm{lb})(20\ \mathrm{ft}) = 2000\,\mathrm{ft\text{-}lb}.$$

This was easy because the force was constant througout the problem (the piano always weighed $100\ \mathrm{lb}$). It is when the force varies (as in the next example) that we need our calculus skills and mastery of integrals.

Example 44. As before, we wish to lift a $100\ \mathrm{lb}$ piano from the ground to the top of a $20\ \mathrm{ft}$ building. We are doing so by standing on the roof and pulling it up using a rope. However, this time, we are using a rather heavy rope weighing $0.1\ \mathrm{lb/ft}$ and want to take that into account (just pulling up the rope, dangling to the ground, would require some work).

Think about the moment when the piano is $x\ \mathrm{ft}$ off the ground (and we want to pull it up by $\mathrm{d}x\ \mathrm{ft}$):

- We still need to pull up $20 - x\ \mathrm{ft}$.

  So, at that moment, the weight (piano plus rope) to be pulled up is $100 + 0.1(20 - x)\ \mathrm{lb}$.
- Hence, to pull up the piano by a tiny amount of $\mathrm{d}x$ feet, the amount of work needed is (roughly) $[100 + 0.1(20 - x)]\mathrm{d}x$ pound.

  [Assuming that $\mathrm{d}x$ is very small, the change in weight is insignificant, so that we can use $W = Fd$.]

To get the total amount of work (in $\mathrm{ft\text{-}lb}$), we need to "add" up these small contributions from $x = 0$ to $x = 20$:

$$\mathrm{work} = \int_0^{20} [100 + 0.1(20 - x)]\mathrm{d}x.$$

It only remains to calculate this integral (which is very simple in this case):

$$\mathrm{work} = \int_0^{20} [102 - 0.1x]\mathrm{d}x = \left[102x - \frac{0.1}{2}x^2\right]_0^{20} = 2020\ \mathrm{ft\text{-}lb}.$$

Comment. In this simple example, you can get away with not computing any integrals by arguing as follows: the rope weighs a total of $20 \cdot 0.1 = 2\ \mathrm{lb}$. On average, we need to lift it $10\ \mathrm{ft}$ so that the total amount of work needed to pull up the rope is $2\ \mathrm{lb} \cdot 10\ \mathrm{ft} = 20\,\mathrm{ft\text{-}lb}$. Added to the work required for just the piano ($2000\,\mathrm{ft\text{-}lb}$, see previous example), we get the total that we just computed.

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Example 45. A conical container of radius 10 ft and height 30 ft is completely filled with water (the tip of the cone is at the bottom). How much work will it take to pump the water to a level of 2 ft above the cone's rim? [Water weighs 62.4 lb/ft^3.]

Solution. Make a sketch! Let us denote with y (in ft) the vertical position in such a way that the tip is at $y=0$ and the rim is at $y=30$. We need to pump the water to the level $y=30+2=32$.

We consider a horizontal slice at height y and thickness $\mathrm{d}y$:

- This slice needs to be lifted up $(30-y)+2=32-y$ (ft).
  (If we considered vertical slices, we could not make such a statement and therefore would be stuck here.)
- The slice is (almost) a disk with radius $r=\frac{10}{30}y=\frac{y}{3}$. Hence, its volume is $\pi r^2\,\mathrm{d}y=\frac{\pi}{9}y^2\mathrm{d}y$ (ft^3).
- The weight of the slice is $62.4\frac{\pi}{9}y^2\,\mathrm{d}y$ (lb) and it needs to be lifted up $32-y$ (ft). That takes work of $(32-y)\cdot 62.4\frac{\pi}{9}y^2\mathrm{d}y$ (ft-lb).
- "Adding" up, the total work required is

$$\int_0^{30}(32-y)\cdot 62.4\frac{\pi}{9}y^2\mathrm{d}y=\ldots\approx 1.86\cdot 10^6 \text{ ft-lb}.$$

Comment. Note that we rounded the answer to 3 significant digits, because we cannot expect more precision given that the weight of water per ft^3 is only given to us to 3 digits.

Example 46. Repeat Example 45 if...

(a) ...the container is not completely filled with water but only to a height of 15 ft.

(b) ...we stop pumping once the container is filled with water to a height of 15 ft.

Can you predict in which case the work required is larger?

Solution.

(a) In that case, the total work required is

$$\int_0^{15}(32-y)\cdot 62.4\frac{\pi}{9}y^2\mathrm{d}y=\ldots\approx 508{,}000 \text{ ft-lb}.$$

(b) In that case, the total work required is

$$\int_{15}^{30}(32-y)\cdot 62.4\frac{\pi}{9}y^2\mathrm{d}y=\ldots\approx 1.35\cdot 10^6 \text{ ft-lb}.$$

Comment. Note that the sum of these two is again $1.86\cdot 10^6 \text{ ft-lb}$. Think about why that makes perfect sense!