Example 47. We want to "lift" a $1600\ \mathrm{kg}$ satellite from the ground into orbit, $20{,}000\ \mathrm{km}$ above the surface. Let us compute the theoretic amount of work required to do so.

Context. These are actually typical values for a GPS satellite. For comparison, the ISS has an average altitude about $400\ \mathrm{km}$, while the moon is about $384{,}000\ \mathrm{km}$ away. Light travels at the speed of about $300{,}000\ \mathrm{km/s}$ (sound only at about $340\ \mathrm{m/s}$).

Solution. First, let us gather the necessary background information:

- Initially, the satellite is sitting on the surface, about $d_1 = 6371\ \mathrm{km}$ from the center of earth (for gravitation, earth behaves like all its mass is concentrated at its center).

  The goal is to bring the satellite to a distance $d_2 = 26{,}371\ \mathrm{km}$ from the center of earth.
- The mass of the earth is about $m_E = 5.972 \cdot 10^{24}\ \mathrm{kg}$. The mass of our satellite is $m_S = 1600\ \mathrm{kg}$.
- The physical law of attraction is $F = G\frac{m_E m_S}{d^2}$.

  It tells us the force of attraction between two masses (here, the satellite and earth) that are at distance d. Here, $G = 6.674 \cdot 10^{-11}\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$ is the gravitational constant.

  Comment. Note that this force is not constant in our problem: the values of d range from $d = d_1$ to $d = d_2$.

Make a sketch! Similar to our approach Example 44, we now think about the moment when the satellite is at distance x from the center of the earth and about the amount of work needed to lift it by $\mathrm{d}x$.

- At that moment, the gravitational force is $G\frac{m_S m_E}{x^2}$.
- To lift up the satellite by a tiny amount of $\mathrm{d}x$, the amount of work needed is (roughly) $G\frac{m_S m_E}{x^2}\mathrm{d}x$ (force times distance).

Hence, the total amount of work is

$$\text{work} = \int_{d_1}^{d_2} G\frac{m_S m_E}{x^2}\mathrm{d}x = G m_S m_E \int_{d_1}^{d_2} \frac{\mathrm{d}x}{x^2} = G m_S m_E \left[-\frac{1}{x}\right]_{d_1}^{d_2} = G m_S m_E \left(\frac{1}{d_1} - \frac{1}{d_2}\right).$$

Plugging in our values for G, m_S, m_E, d_1, d_2, we find

$$\text{work} = (6.674 \cdot 10^{-11}) \cdot (1600) \cdot (5.972 \cdot 10^{24}) \left(\frac{1}{6371000} - \frac{1}{26371000}\right) \approx 7.59 \cdot 10^{10}\ \text{joule}.$$

Comment. You could also let x be the distance from the surface. That works as well if you adjust things accordingly. (Do that as an exercise!)

Comment. If you choose $d = 6371\ \mathrm{km}$ in the law of attraction, set one mass to be the mass $m_E = 5.972 \cdot 10^{24}\ \mathrm{kg}$ of earth and the other to be $1\ \mathrm{kg}$, then the resulting force is $F = 6.674 \cdot 10^{-11} \cdot \frac{5.972 \cdot 10^{24}}{(6.371 \cdot 10^6)^2} \approx 9.820\ \mathrm{N}$, which you have surely met in other class before. Note that this force is an approximation and depends on the exact elevation (earth is not perfectly round). Usually, the value $9.81\ \mathrm{N}$ is used.

Example 48.

- What happens when we take the **limit** $d_2 \to \infty$ in the previous example? What does that mean physically?
- How does the previous problem change if the physical law of attraction was $F = G\frac{m_S m_E}{d}$? What happens now when we take the limit $d_2 \to \infty$?

Example 49. (extra) The Pyramid of Cheops, built about 2560 BC, has been the tallest man-made structure in the world for over 3800 years. The pyramid was built to a height of 146m. Its base is a square with each side 230m in length.

(a) The pyramid is made out of limestone (1 cubicmeter of limestone has a mass of 2.3 tonnes). Assuming that the pyramid is solid, compute its mass (in tonnes) by using an integral.

(b) What was the mass of the pyramid when it was built to half its final height?

(c) The limestone blocks used usually have a mass of about 2.5 tonnes each. (Roughly) how many limestone blocks does the pyramid consist of?

(d) Compute the (theoretical) total amount of work (in joule) that was required in lifting all the blocks from the ground to their final position.

(e) Of course, the actual amount of work required was much higher; assume it was 50 times as high as the theoretical amount you just calculated. Further, assume that an Egyptian worker could perform work of about 2000 kilojoules per day. Based on these numbers and no holidays, how many workers would have been needed to construct the Pyramid of Cheops in 20 years?
Comment. That's just for the building part! Sourcing and transport of the blocks and all other stuff not included ... For perspective, assume a person consumes 2000 calories a day. That actually means $2000\text{ kcal} \approx 8400\text{ kJ}$, and that is an upper limit on how much work they can perform on a daily basis.

Solution.

(a) Let us denote with x (in m) the vertical position in such a way that the tip is at $x=0$ and the base is at $x=146$. That way, the cross-section at x has a width of $\frac{230}{146}x$ (so that the width is 0 when $x=0$, and the width is 230 when $x=146$). The volume of the pyramid is

$$\int_0^{146}\left(\frac{230}{146}x\right)^2 \mathrm{d}x = \left(\frac{230}{146}\right)^2 \frac{1}{3}x^3\bigg|_0^{146} = \frac{1}{3}230^2\cdot 146 \approx 2.57\cdot 10^6 \quad \mathrm{m}^3$$

and hence its weight is

$$\frac{1}{3}230^2\cdot 146\cdot 2.3 \approx 5.921\cdot 10^6 \quad \mathrm{t}.$$

(b) The volume of the pyramid was

$$\int_{73}^{146}\left(\frac{230}{146}x\right)^2 \mathrm{d}x = \left(\frac{230}{146}\right)^2 \frac{1}{3}x^3\bigg|_{73}^{146} = \frac{7}{24}230^2\cdot 146 \quad \mathrm{m}^3$$

and hence its weight about 5.181 million tonnes.

(c) About $\frac{5.921\cdot 10^6}{2.5} \approx 2.369\cdot 10^6$, or 2.4 million, blocks of limestone.

(d) Recall that on earth's surface, one kg weighs about $9.81\ \mathrm{N}$. The work therefore is

$$\int_0^{146}\left(\frac{230}{146}x\right)^2 (146-x)2300\cdot 9.81\mathrm{d}x \approx 2.120\cdot 10^{12} \quad \mathrm{J}.$$

(e) The building takes

$$\frac{50\cdot 2.120\cdot 10^{12}}{2{,}000{,}000} \approx 5.30\cdot 10^7$$

days of a man's work. To complete the work in 20 years

$$\frac{5.30\cdot 10^7}{20\cdot 365} \approx 7260$$

workers would be needed.

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