Review. (from midterm exam) Here are two ways to determine $\int \frac{\mathrm{d}x}{2x}$. Which is correct?

(a) $\int \frac{\mathrm{d}x}{2x} = \frac{1}{2}\int \frac{\mathrm{d}x}{x} = \frac{1}{2}\ln|x| + C$

(b) We substitute $u = 2x$ (so that $\mathrm{d}u = 2\mathrm{d}x$) to get:

$$\int \frac{\mathrm{d}x}{2x} = \int \frac{\frac{1}{2}\mathrm{d}u}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x| + C$$

Solution. Both are correct! The answers look different but they only differ by a constant because

$$\frac{1}{2}\ln|2x| = \frac{1}{2}\ln(2|x|) = \frac{1}{2}(\ln(2) + \ln|x|).$$

Integration by parts

If we integrate both sides of the product rule, we obtain the following:

$$\begin{aligned} (fg)' &= f'g + fg' \\ \overset{\text{antiderivative}}{\rightsquigarrow} \quad f(x)g(x) &= \int f'(x)g(x)\mathrm{d}x + \int f(x)g'(x)\mathrm{d}x \end{aligned}$$

If we then solve for one of the two integrals, we get:

(integration by parts)

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

The following shorthand is very common as well:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Here, $u = f(x)$, $v = g(x)$ so that $\mathrm{d}u = f'(x)\mathrm{d}x$ and $\mathrm{d}v = g'(x)\mathrm{d}x$.

Example 83. Determine $\int x\cos(x)\,\mathrm{d}x$.

Solution. We choose $f(x) = x$ and $g'(x) = \cos(x)$, so that $g(x) = \sin(x)$ (note that we are free to choose the simplest antiderivative for $g(x)$), to get

$$\int x\cos(x)\,\mathrm{d}x = x\sin(x) - \int 1\cdot\sin(x)\,\mathrm{d}x = x\sin(x) + \cos(x) + C.$$

Example 84. Determine $\int x e^x\,\mathrm{d}x$.

Solution. We choose $f(x) = x$ and $g'(x) = e^x$, so that $g(x) = e^x$, to get

$$\int x e^x\,\mathrm{d}x = x e^x - \int 1\cdot e^x\,\mathrm{d}x = x e^x - e^x + C = (x-1)e^x + C.$$

Armin Straub
straub@southalabama.edu

Example 85. Determine $\int \ln(x)\,\mathrm{d}x$.

Solution. We choose $f(x) = \ln(x)$ and $g'(x) = 1$, so that $g(x) = x$, to get

$$\int \ln(x) \cdot 1\,\mathrm{d}x = \ln(x) \cdot x - \int \frac{1}{x} \cdot x\,\mathrm{d}x = x\ln(x) - x + C.$$

Example 86. Substitute $u = \ln(x)$ in the previous integral. What do you get?

Solution. If $u = \ln(x)$ then $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ so that $\mathrm{d}x = x\,\mathrm{d}u = e^u\mathrm{d}u$ (in the last step, we used that $x = e^u$).

We therefore get $\int \ln(x)\,\mathrm{d}x = \int ue^u\,\mathrm{d}u$.

By Example 84 we know $\int ue^u\,\mathrm{d}u = (u-1)e^u + C$ so that

$$\int \ln(x)\,\mathrm{d}x = \int ue^u\,\mathrm{d}u = (u-1)e^u + C = (\ln(x) - 1)e^{\ln(x)} + C = (\ln(x) - 1)x + C,$$

which matches what we obtained in Example 85.

Comment. We can also start by writing $x = e^u$ so that we immediately get $\mathrm{d}x = e^u\mathrm{d}u$. It depends on the integral, which of the two approaches is algebraically simpler.