Review. By integrating the product rule, we get the formula for integration by parts:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x.$$

Example 87. (again) Determine $\int x e^x \, \mathrm{d}x$.

- To evaluate the integral, we should choose $f(x) = x$ and $g'(x) = e^x$, because $f'(x)$ becomes easier while $g(x)$ stays the same. Do it!
- If, on the other hand, we decide to choose $f(x) = e^x$ and $g'(x) = x$, then we obtain

  $$\int x e^x \, \mathrm{d}x = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x \, \mathrm{d}x.$$

  While certainly correct, we actually ended up with a more difficult integral.

  [On the other hand, because we know $\int x e^x \, \mathrm{d}x$, this means we now also know $\int x^2 e^x \, \mathrm{d}x$.]

Example 88. Determine $\int_0^1 x^2 e^{3x} \, \mathrm{d}x$.

Solution. We do integration by parts with $f(x) = x^2$ and $g'(x) = e^{3x}$ so that $f'(x) = 2x$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x^2 e^{3x} \, \mathrm{d}x = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x e^{3x} \, \mathrm{d}x.$$

We now do integration by parts again with $f(x) = x$ and $g'(x) = e^{3x}$ so that $f'(x) = 1$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x e^{3x} \, \mathrm{d}x = \frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x} \, \mathrm{d}x = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x}.$$

Taken together, this means

$$\int x^2 e^{3x} \, \mathrm{d}x = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x}\right) = \left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x}.$$

In particular,

$$\int_0^1 x^2 e^{3x} \, \mathrm{d}x = \left[\left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x}\right]_0^1 = \left(\frac{1}{3} - \frac{2}{9} + \frac{2}{27}\right)e^3 - \frac{2}{27} = \frac{5}{27}e^3 - \frac{2}{27}.$$

Alternatively. While doing integration by parts, we can carry the bounds along. For instance, for the first step,

$$\int_0^1 x^2 e^{3x} \, \mathrm{d}x = \left[\frac{1}{3}x^2 e^{3x}\right]_0^1 - \frac{2}{3}\int_0^1 x e^{3x} \, \mathrm{d}x = \frac{1}{3}e^3 - \frac{2}{3}\int_0^1 x e^{3x} \, \mathrm{d}x.$$

For practice, do the second step likewise to get the same final answer as before!

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Example 89. Determine $\int e^x \cos(x)\,\mathrm{d}x$.

Solution. We will need to integrate by parts twice. First, let $f(x) = \cos(x)$ and $g'(x) = e^x$ so that $f'(x) = -\sin(x)$ and $g(x) = e^x$ (the other way around works as well—see below!) to get

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \cos(x) + \int e^x \sin(x)\,\mathrm{d}x.$$

The new integral is of the same level of difficulty, so it might seem like we haven't gained anything. But don't give up yet! Instead, integrate by parts again with $f(x) = \sin(x)$ and $g'(x) = e^x$ to arrive at

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \cos(x) + \int e^x \sin(x)\,\mathrm{d}x = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\,\mathrm{d}x.$$

We can now solve for $\int e^x \cos(x)\,\mathrm{d}x$ and find $\int e^x \cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x \sin(x) + e^x \cos(x))$.

Solution. (variation) We proceed as before but now let $f(x) = e^x$ and $g'(x) = \cos(x)$ to get

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{d}x.$$

Again, we once more integrate by parts: choosing $f(x) = e^x$ and $g'(x) = \sin(x)$, we arrive at

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{d}x = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,\mathrm{d}x.$$

As before, we can then solve for $\int e^x \cos(x)\,\mathrm{d}x$ to find $\int e^x \cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x \sin(x) + e^x \cos(x))$.

We next discuss integrals of products of trig functions. The following is an example that we are already familiar with:

Example 90. (review/preview) $\int \sin^{\lambda}(x)\cos(x)\,\mathrm{d}x =$ (with $\lambda \neq -1$)

Solution. We substitute $u = \sin(x)$, because then $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin^{\lambda}(x)\cos(x)\,\mathrm{d}x = \int u^{\lambda}\mathrm{d}u = \frac{1}{\lambda+1}u^{\lambda+1} + C = \frac{\sin^{\lambda+1}(x)}{\lambda+1} + C.$$