Review. Recall that $\mathrm{d}g(x) = g'(x)\mathrm{d}x$ (since $\frac{\mathrm{d}}{\mathrm{d}x}g(x) = g'(x)$). Integration by parts therefore is often written as

$$\int f(x)\mathrm{d}g(x) = f(x)g(x) - \int g(x)\mathrm{d}f(x), \quad \text{or} \quad \int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

In the latter short form, we have set $u = f(x)$ and $v = g(x)$.

Trigonometric integrals

The following example illustrates that we somtimes have choices when integrating:

Example 91. $\int \sin(x)\cos(x)\,\mathrm{d}x$

Solution. (integration by parts) Integrating by parts with $f(x) = \sin(x)$, $g'(x) = \cos(x)$, $g(x) = \sin(x)$, we get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \sin^2(x) - \int \cos(x)\sin(x)\,\mathrm{d}x,$$

from which we conclude that $\int \sin(x)\cos(x)\,\mathrm{d}x = \frac{1}{2}\sin^2(x) + C$.

Solution. (substitution) Substitute $u = \sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

Solution. (trig identity) Since $\sin(2x) = 2\cos(x)\sin(x)$, we have

$$\int \sin(x)\cos(x)\,\mathrm{d}x = \frac{1}{2}\int \sin(2x)\mathrm{d}x = -\frac{1}{4}\cos(2x) + C.$$

Important comment. Note that $\frac{1}{2}\sin^2(x) \neq -\frac{1}{4}\cos(2x)$ (for instance, plug in $x = 0$ to see that). However, the two functions are indeed equal up to a constant (namely, $\frac{1}{2}\sin^2(x) = -\frac{1}{4}\cos(2x) + \frac{1}{4}$) as we can see from the trig identity $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

Example 92. $\int \sin^m(x)\cos^3(x)\,\mathrm{d}x$ (with $m \neq -1, -3$)

Solution. We substitute $u = \sin(x)$, because $\mathrm{d}u = \cos(x)\,\mathrm{d}x$, to get

$$\begin{aligned}\int \sin^m(x)\cos^3(x)\,\mathrm{d}x &= \int u^m\cos^2(x)\,\mathrm{d}u = \int u^m(1 - \sin^2(x))\,\mathrm{d}u = \int u^m(1 - u^2)\mathrm{d}u \\ &= \frac{u^{m+1}}{m+1} - \frac{u^{m+3}}{m+3} + C = \frac{\sin^{m+1}(x)}{m+1} - \frac{\sin^{m+3}(x)}{m+3} + C.\end{aligned}$$

Armin Straub
straub@southalabama.edu

The strategy in the previous problem works whenever we have an odd power of cosine:

Example 93. Describe how we can determine $\int \sin^m(x)\cos^{2k+1}(x)\,\mathrm{d}x$.

Solution. Again, we substitute $u=\sin(x)$, because $\mathrm{d}u=\cos(x)\,\mathrm{d}x$, to get

$$\int \sin^m(x)\cos^{2k+1}(x)\,\mathrm{d}x = \int u^m\cos^{2k}(x)\,\mathrm{d}u = \int u^m(1-\sin^2(x))^k\,\mathrm{d}u = \int u^m(1-u^2)^k\mathrm{d}u.$$

For a given integer $k\geqslant 0$, we can now multiply out the term $(1-u^2)^k$. Each resulting term (after multiplying with u^m) can then be integrated using the power rule (as in the previous example).

Extrapolating this strategy, we can integrate the following products of trigonometric function:

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with $n=2k+1$ odd, can be evaluated by substituting $u=\sin(x)$.

  See previous example!

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with m odd, can be likewise evaluated by substituting $u=\cos(x)$.

- $\int \sin^m(x)\cos^n(x)\,\mathrm{d}x$, with both m,n even, can be reduced via

  $$\sin^2(x)=\frac{1-\cos(2x)}{2}, \quad \cos^2(x)=\frac{1+\cos(2x)}{2}.$$

  [Then, multiply out the integrand. The resulting integrals have smaller exponents, and we (recursively) apply our strategy to each of them (if the $2x$ bothers you, substitute $u=2x$).]

Example 94. Determine $\int \cos^2(x)\mathrm{d}x$.

Solution. (trig identity) Since both exponents are even (the exponent of $\sin(x)$ is 0, which is even), we use the trig identity $\cos^2(x)=\frac{1+\cos(2x)}{2}$:

$$\int \cos^2(x)\mathrm{d}x = \int \frac{1+\cos(2x)}{2}\mathrm{d}x = \frac{1}{2}x+\frac{1}{4}\sin(2x)+C.$$

Solution. (integration by parts—only for practice) We choose $f(x)=\cos(x)$ and $g'(x)=\cos(x)$ (so that $g(x)=\sin(x)$) to get

$$\int \cos^2(x)\mathrm{d}x = \cos(x)\sin(x)+\int \sin^2(x)\mathrm{d}x = \cos(x)\sin(x)+\int (1-\cos^2(x))\mathrm{d}x.$$

Note that our integral appears on both sides. Solving for it, we conclude that

$$\int \cos^2(x)\mathrm{d}x = \frac{1}{2}(\cos(x)\sin(x)+x)+C.$$

Our final answer looks different at first glance but is the same because $\sin(2x)=2\cos(x)\sin(x)$.

Example 95. Determine $\int \cos^2(x)\sin^2(x)\mathrm{d}x$.

Solution. Since both exponents are even, we use the trig identities $\cos^2(x) = \frac{1+\cos(2x)}{2}$, $\sin^2(x) = \frac{1-\cos(2x)}{2}$:

$$\int \cos^2(x)\sin^2(x)\mathrm{d}x = \int \frac{1+\cos(2x)}{2}\cdot\frac{1-\cos(2x)}{2}\mathrm{d}x = \frac{1}{4}\int (1-\cos^2(2x))\mathrm{d}x = \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x)\mathrm{d}x.$$

We now use $\cos^2(x) = \frac{1+\cos(2x)}{2}$ again (or we could use the previous example) to find

$$\int \cos^2(2x)\mathrm{d}x = \int \frac{1+\cos(4x)}{2}\mathrm{d}x = \frac{1}{2}x + \frac{1}{8}\sin(4x) + B.$$

Combined, we have (we rename the constant of integration to absorb the factor of $-1/4$)

$$\int \cos^2(x)\sin^2(x)\mathrm{d}x = \frac{1}{4}x - \frac{1}{4}\left(\frac{1}{2}x + \frac{1}{8}\sin(4x)\right) + C = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C.$$