Example 99. Determine $\int \sqrt{1-x^2}\,\mathrm{d}x$.

Solution. We substitute $x=\sin\theta$ (with $\theta\in(-\pi/2,\pi/2)$ so that $\theta=\arcsin(x)$) because then $1-x^2=\cos^2\theta$. Since $\mathrm{d}x=\cos\theta\,\mathrm{d}\theta$, we find

$$\int \sqrt{1-x^2}\,\mathrm{d}x=\int \cos^2\theta\,\mathrm{d}\theta=\ldots\text{by parts}\ldots=\frac{1}{2}(\cos(\theta)\sin(\theta)+\theta)+C=\frac{1}{2}\left(x\sqrt{1-x^2}+\arcsin x\right)+C.$$

See Example 94 for the integration by parts. In the final step, we used $\cos\theta=\sqrt{1-\sin^2\theta}=\sqrt{1-x^2}$ (instead of $\cos(\arcsin(x))$).

Comment. Note that $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ is the area of a quarter of the unit circle (and so has to be $\pi/4$). Using the antiderivative we just computed, we indeed find (since $\sin(\pi/2)=1$ we have $\arcsin(1)=\pi/2$)

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x=\left[\frac{\arcsin x+x\sqrt{1-x^2}}{2}\right]_0^1=\frac{\frac{\pi}{2}+0}{2}-\frac{0+0}{2}=\frac{\pi}{4}.$$

Example 100. Determine $\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t$.

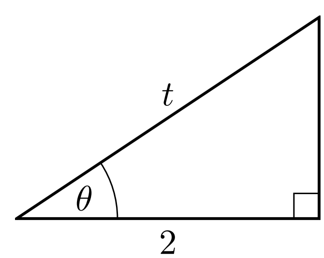
Solution. We substitute $t=2\sec\theta$ because then $t^2-4=4(\sec^2\theta-1)=4\tan^2\theta$.

Since $\frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}\theta}2\sec\theta=2\sec\theta\tan\theta$ (you can work this out from $\sec\theta=\frac{1}{\cos\theta}$), we get

$$\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t=\int \frac{1}{4\sec^2\theta\sqrt{4\tan^2\theta}}2\sec\theta\tan\theta\mathrm{d}\theta=\frac{1}{4}\int\frac{1}{\sec\theta}\mathrm{d}\theta=\frac{1}{4}\int\cos\theta\mathrm{d}\theta=\frac{1}{4}\sin\theta+C.$$

Our final step consists in simplifying $\sin\theta$ given that $t=2\sec\theta$.

For this, draw a right-angled triangle with angle θ. To encode the relationship $\sec\theta=\frac{\text{hyp}}{\text{adj}}=\frac{t}{2}$, we assign the hypothenuse length t and the adjacent side length 2 as in the diagram to the right.



By Pythagoras, the opposite side then has length $\sqrt{t^2-4}$. It follows that

$$\sin\theta=\frac{\text{opp}}{\text{hyp}}=\frac{\sqrt{t^2-4}}{t}.$$

Overall, we have therefore found that

$$\int \frac{1}{t^2\sqrt{t^2-4}}\,\mathrm{d}t=\frac{\sqrt{t^2-4}}{4t}+C.$$

Example 101. Determine $\int \frac{1}{1+x^2}\,\mathrm{d}x$.

Solution. Of course, we already know that $\int \frac{1}{1+x^2}\mathrm{d}x=\arctan(x)+C$. On the other hand, in the alternative solution below, we pretend that we didn't.

Solution. We substitute $x=\tan\theta$ because then $1+x^2=\sec^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta=\sec^2\theta$, we get

$$\int \frac{1}{1+x^2}\mathrm{d}x=\int\frac{\sec^2\theta\mathrm{d}\theta}{\sec^2\theta}=\int\mathrm{d}\theta=\theta+C=\arctan(x)+C.$$

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Example 102. Determine $\int \frac{1}{(1+x^2)^2}\mathrm{d}x$. [That's an integral we care about for partial fractions!]

Solution. We substitute $x = \tan\theta$ because then $1 + x^2 = \sec^2\theta$. Since $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta = \sec^2\theta$, we get

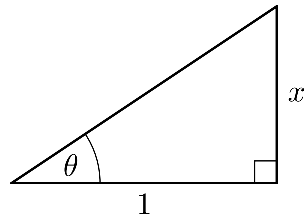
$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x = \int \frac{\sec^2\theta\mathrm{d}\theta}{(\sec^2\theta)^2} = \int \frac{\mathrm{d}\theta}{\sec^2\theta} = \int \cos^2\theta\mathrm{d}\theta.$$

From Example 94, we know that

$$\int \cos^2\theta\mathrm{d}\theta = \frac{1}{2}(\cos(\theta)\sin(\theta) + \theta) + C.$$

After replacing $\theta = \arctan(x)$, we could stop here, except that our answer can be considerable simplified!

For this, draw a right-angled triangle with angle θ. To encode the relationship $\tan\theta = \frac{\mathrm{opp}}{\mathrm{adj}} = x$, we assign the opposite side length x and the adjacent side length 1 as in the diagram to the right.



By Pythagoras, the hypothenuse then has length $\sqrt{1+x^2}$. It follows that

$$\cos\theta = \frac{\mathrm{adj}}{\mathrm{hyp}} = \frac{1}{\sqrt{1+x^2}}, \quad \sin\theta = \frac{\mathrm{opp}}{\mathrm{hyp}} = \frac{x}{\sqrt{1+x^2}}.$$

Hence $\cos(\theta)\sin(\theta) = \frac{x}{1+x^2}$ so that, combined, we get

$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x = \frac{1}{2}(\cos(\theta)\sin(\theta) + \theta) + C = \frac{1}{2}\left[\frac{x}{1+x^2} + \arctan(x)\right] + C.$$

Comment. We just showed that, for instance, $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$.