Example 108. Evaluate $\int \frac{x^4+3x^3+1}{x^3-x}\,\mathrm{d}x$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get
$$\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x^3-x}.$$
- Factor the denominator: $x^3-x = x(x^2-1) = x(x-1)(x+1)$.
- $\frac{x^4+3x^3+1}{x^3-x} = x+3+\frac{x^2+3x+1}{x(x-1)(x+1)}$
- By partial fractions $\frac{x^2+3x+1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$ for certain numbers A, B, C.
  - We multiply both sides with $x(x-1)(x+1)$ to clear denominators:
  $$x^2+3x+1 = (x-1)(x+1)A + x(x+1)B + x(x-1)C$$
  - Set $x=0$ to get $1=-A$ so that $A=-1$.
    Set $x=1$ to get $5=2B$ so that $B=\frac{5}{2}$.
    Set $x=-1$ to get $-1=2C$ so that $C=-\frac{1}{2}$.

Therefore,

$$\begin{aligned}\int \frac{x^4+3x^3+1}{x^3-x}\,\mathrm{d}x &= \int \left(x+3-\frac{1}{x}+\frac{5/2}{x-1}-\frac{1/2}{x+1}\right)\mathrm{d}x \\ &= \frac{1}{2}x^2+3x-\ln|x|+\frac{5}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|.\end{aligned}$$

Example 109. Determine the shape (but not the exact numbers involved) of the partial fraction decomposition of the following rational functions.

(a) $\frac{x^2-2}{x^4-x^2}$

(b) $\frac{x^7-2}{x^4-x^2}$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2}$

Solution.

(a) $\frac{x^2-2}{x^4-x^2} = \frac{x^2-2}{x^2(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$

(b) Note that in this case, we need to do long division first. Since $x^7/x^4 = x^3$, the result is of the form Ax^3+Bx^2+Cx+D with some remainder that still needs to be divided by x^4-x^2. Hence:
$\frac{x^7-2}{x^4-x^2} = \frac{x^7-2}{x^2(x-1)(x+1)} = Ax^3+Bx^2+Cx+D+\frac{E}{x}+\frac{F}{x^2}+\frac{G}{x-1}+\frac{H}{x+1}$

(c) $\frac{x^3-7x+1}{x^2(x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}$

(d) $\frac{x^2+5}{(x+2)^3(x^2+1)^2} = \frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{(x+2)^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}$

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Example 110. Evaluate $\int \frac{x^7+3x+1}{x^4+x^2}\mathrm{d}x$.

Solution.

- Since the degree of the numerator is not less than the degree of the denominator, we first perform long division. In this case, we get

$$\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{x^3+3x+1}{x^4+x^2}.$$

- For the remainder part, partial fractions now tells us its decomposed shape:

$$\frac{x^3+3x+1}{x^4+x^2} = \frac{x^3+3x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

  - We multiply both sides with $x^2(x^2+1)$ to clear denominators:

$$x^3+3x+1 = x(x^2+1)A + (x^2+1)B + x^2(Cx+D)$$

  - We can now compare the coefficients of $x^3, x^2, x, 1$ on both sides.
    Coefficients of x^3: $1 = A + C$
    Coefficients of x^2: $0 = B + D$
    Coefficients of x: $3 = A$
    Coefficients of 1: $1 = B$.
    Hence, $A = 3$, $B = 1$, $C = 1 - A = -2$, $D = -B = -1$.
    Note. By coefficient of 1 we mean the constant terms of the polynomials (the stuff without any x).
    Alternatively. We can also plug in values for x to get equations in A, B, C, D. Unfortunately, our only "magic" choice is $x = 0$. This gives $B = 1$. Instead of plugging in random values for x (we could do that!) we can then subtract the $(x^2+1)B$ from both sides and divide by x to get the simpler $x^2 - x + 3 = (x^2+1)A + x(Cx+D)$. Then we can again set $x = 0$ to find $3 = A$. Finish this for practice!

  - Consequently: $\frac{x^7+3x+1}{x^4+x^2} = x^3 - x + \frac{3}{x} + \frac{1}{x^2} + \frac{-2x-1}{x^2+1}$.

Finally, we can integrate to find:

$$\begin{aligned}\int \frac{x^7+3x+1}{x^4+x^2}\mathrm{d}x &= \int \left(x^3 - x + \frac{3}{x} + \frac{1}{x^2} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right)\mathrm{d}x \\ &= \frac{1}{4}x^4 - \frac{1}{2}x^2 + 3\ln|x| - \frac{1}{x} - \ln(x^2+1) - \arctan(x) + C\end{aligned}$$

Here, we computed $\int \frac{2x}{x^2+1}\mathrm{d}x = \ln(x^2+1) + C$ by substituting $u = x^2+1$. Do it!

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