Review. The following forms are indeterminate: "$\frac{\infty}{\infty}$", "$\frac{0}{0}$", "$0 \cdot \infty$", "∞^0", "1^∞", "0^0". By applying $\ln$ in the last three cases, we can always write these as "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$", so that we can apply L'Hospital.

Example 123. Determine the following limits:

(a) $\lim_{n\to\infty} \frac{\ln n}{n}$

(b) $\lim_{n\to\infty} n^{1/n}$

(c) $\lim_{n\to\infty} \sqrt[n]{\pi^2}$

(d) $\lim_{n\to\infty} \sqrt[n]{n^2}$

(e) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n$

Solution.

(a) $\lim_{n\to\infty} \frac{\ln n}{n} \overset{\text{LH}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$

We were able to use L'Hospital because the limit was of the form "$\frac{\infty}{\infty}$".

(b) $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp(\ln(n^{1/n})) = \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln(n)\right) = \exp(0) = 1$

Note that we used the limit from the previous part.

Comment. We wrote $\exp(x) = e^x$ simply to avoid using exponents for typographical reasons.

(c) $\lim_{n\to\infty} \sqrt[n]{\pi^2} = \lim_{n\to\infty} \pi^{2/n} = \pi^0 = 1$

(d) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} \exp(\ln(n^{2/n})) = \lim_{n\to\infty} \exp\left(\frac{2}{n}\ln(n)\right) = \exp(0) = 1$

(e) $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} \exp\left(\ln\left(\left(1+\frac{2}{n}\right)^n\right)\right) = \lim_{n\to\infty} \exp\left(n\ln\left(1+\frac{2}{n}\right)\right) = e^2$

In the final step, we used that $\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{2}{n}} \cdot \left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}} = 2.$

Comment. More generally, $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for any x.

The following is a precise definition of the limit of a sequence:

Definition 124. $\lim_{n\to\infty} a_n = L$ means that:
for every $\varepsilon > 0$ there is a value N such that, for all $n > N$, $|a_n - L| < \varepsilon$.

Here are a few basic facts about limits:

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$
- If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ then $\lim_{n\to\infty} (a_n + b_n) = A + B$ and $\lim_{n\to\infty} (a_n b_n) = AB$.
- If $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} f(a_n) = f(A)$ provided that $f(x)$ is continuous at A.

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Example 125. $\lim_{n\to\infty} x^n = \begin{cases} \infty, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ 0, & \text{if } -1 < x < 1, \\ \text{does not exist}, & \text{if } x \leqslant -1. \end{cases}$

If you think of a representative case for each situation, then this (important!) example becomes very natural:

- $\lim_{n\to\infty} 2^n =$
- $\lim_{n\to\infty} 1^n =$
- $\lim_{n\to\infty} (1/2)^n =$
- $\lim_{n\to\infty} (-1/2)^n =$
- $\lim_{n\to\infty} (-1)^n =$
- $\lim_{n\to\infty} (-2)^n =$

The geometric sum

(geometric sum)

$$\sum_{n=0}^{M} x^n = 1 + x + x^2 + \ldots + x^M = \frac{1 - x^{M+1}}{1 - x}$$

Why? Let us write $S = 1 + x + x^2 + \ldots + x^M$.
Note that $xS = x + x^2 + \ldots + x^M + x^{M+1}$ and that the result has most terms in common with our original sum. In fact, the right-hand side is $S - 1 + x^{M+1}$. This means that

$$xS = S - 1 + x^{M+1}.$$

Solving this for S, we find that $S = \frac{x^{M+1} - 1}{x - 1}$ which is equivalent to the above formula.

Example 126. Determine the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^M}$. What happens as $M \to \infty$?

Solution. This is a geometric sum with $x = \frac{1}{2}$. Thus,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^M} = \frac{1 - \left(\frac{1}{2}\right)^{M+1}}{1 - \frac{1}{2}}.$$

Note that

$$\lim_{M\to\infty} \frac{1 - \left(\frac{1}{2}\right)^{M+1}}{1 - \frac{1}{2}} = \frac{1 - 0}{1 - \frac{1}{2}} = 2.$$